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Flame-acoustic resonance initiated by vortical disturbances

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By adapting the general flame-acoustic interaction theory developed in Wu *et al.* (*J. Fluid Mech.*, vol. 497, 2003, pp. 23–53), a systematic analysis is carried out for the interaction of a stable premixed flame in a duct with vortical disturbances superimposed on the oncoming mixture. A small-amplitude vortical perturbation, assumed to be a convecting gust with a frequency ω, induces a hydrodynamic field in the vicinity of the flame, causing an initially planar flame to wrinkle. The unsteady heat release resulting from the increased surface area of the wrinkling flame then generates a sound wave with frequency 2ω. When 2ω coincides with the natural frequency of an acoustic mode of the duct, a flame-acoustic resonance takes place, through which the flame-induced sound may attain an amplitude sufficiently large to modulate the flame through the unsteady Rayleigh–Taylor effect. A novel evolution system is derived to describe this two-way coupling for two cases: (*a*) a flame with a fixed mean position and (*b*) a moving flame. Numerical solutions show that for (*a*), the mutual flame-acoustic interaction initiates a violent subharmonic parametric instability, and the flame-acoustic system quickly evolves into a fully nonlinear regime, which probably corresponds to a state of self-sustained oscillation. This finding presents a peculiar instability scenario: a small-amplitude vortical perturbation may, by initiating acoustic-flame resonance, completely destabilize an otherwise stable planar flame. For a moving flame, the flame-acoustic resonance is of transient nature. The acoustic pressure gains substantially, but the parametric flame instability is induced only when the vortical disturbance exceeds a finite threshold.

1. Introduction

The influence of external disturbances on premixed flames has been extensively studied. The interest is twofold. The first is associated with modelling the turbulent flame velocity U_T in terms of the ambient flow characteristics, such as the fluctuation intensity u'. Clavin & Williams (1982) suggested that the initial departure from the laminar flame speed U_L may be understood by analysing the response of a flame to small-amplitude vortical disturbances representing weak turbulence present in the oncoming fresh mixture. Since the flame thickness is usually much smaller than the characteristic length scale of flow fluctuations, the flame may be treated as an 'interface' separating the burned and unburned materials. Describing the response of the 'interface', however, requires an analysis of its inner structure, which consists of

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a thin preheat zone and an even thinner reaction zone. Heat release from within the flame induces a non-trivial hydrodynamical field via gas expansion, which means that the propagation of the interface cannot be predicted without knowing its hydrodynamic field. Not surprisingly therefore, only after the asymptotic theory of the flame structure (Matalon & Matkowsky 1982; Pelce & Clavin 1982) had been well established could complete linearized analyses be performed, by Searby & Clavin (1986) and Aldredge & Williams (1991). These authors computed relevant statistical properties of the wrinkled flame and the hydrodynamic field for oncoming disturbances with a given spectrum. The increase of the flame speed over U_L, due to the increased surface area, scales as u'^2 as expected. Since these calculations sought the steady-state response, the flame was assumed to be in the parameter regime in which the hydrodynamic, i.e. Darrieus–Landau (D-L), instability does not arise. Response of weakly unstable flames to moderate level of perturbations has been investigated by using equations of the Michelson–Sivashinsky (M-S) type (see e.g. Cambray & Joulin 1994; D'Angelo, Joulin & Boury 2000; Zaytsev & Bychkov 2002). Propagation of flame fronts through intense disturbances was studied by using the so-called G-equation (e.g. Zhu & Ronney 1994; Aldredge 1996).

The second, and probably more important, motivation of investigating the interaction of external disturbances with a flame is due to its relevance to the large-scale instability of combustion, which generally refers to strong pressure fluctuations of acoustic nature in a chamber (see e.g. Poinsot *et al.* 1987; Yu, Trouve & Daily 1991; Candel 2002). While this instability is essentially a self-excited oscillation, involving a complex interplay among unsteady chemical heat release and transport, hydrodynamics and acoustic modes of the chamber, it was believed that coupling of any two of these multiple physical processes may be isolated and characterized by a transfer function, by studying the response of a flame to suitable externally imposed perturbations (Lieuwen 2003). The expectation is that the full closed-loop coupling may eventually be described by compounding these transfer functions. The problem is of particular relevance for suppressing combustion instability by means of active control (e.g. Candel 2002; Dowling & Morgans 2005), where it is important to know how a flame responds to actuation.

In connection with combustion instability, there have been a great number of investigations of flame response to acoustic disturbances (see e.g. Ducruix *et al.* 2003 and references therein). The flame motion and the resulting heat release have been measured, mostly for conical flames, in order to extract transfer functions relating the response to the forcing (e.g. Ducruix, Durox & Candel 2000; Schuller *et al.* 2002). For weak perturbations, these flames are found to act as a low-pass filter. In the same vein, Baillot, Durox & Prud'homme (1992) studied the flame response to small-amplitude vortical disturbances. In examining the flow field upstream of a conical flame subjected to acoustic excitation, Birbaud, Durox & Candel (2006) detected a significant level of convective vortical fluctuations, suggesting that this kind of disturbance is inevitably present, even if not artificially introduced, and should therefore deserve further investigations.

Semi-empirical kinematic models based on the G-equation have been proposed by Fleifil *et al.* (1996) to describe flame wrinkling caused by perturbations, from which the unsteady heat release can then be estimated. Models of this type have been further extended by Dowling (1999), Schuller, Durox & Candel (2003) and Lieuwen (2005) and were found to give reasonably good predictions. However, with the hydrodynamic and thermal-diffusive aspects of the flame being bypassed, this approach includes only one effect of the perturbation on the flame: the kinematic advection of the flame by the acoustic *velocity*. In reality, acoustic *pressure* influences the burning rate of the flame, and the acoustic pressure gradient, or acceleration, affects the flame dynamics via the Rayleigh–Taylor (R-T) effect. In order to account for these effects, it is necessary to look into the structure of the flame as well as its associated hydrodynamic field (Matalon 2007). The activation-energy-asymptotic (AEA) approach based on large Zeldovich number provides the mathematical tool for pursuing this line of investigation on a first-principle basis (Clavin 1985, 1994).

Effects of acoustic pressures on the burning rate have been analysed by several investigators using AEA. Harten, Kapila & Matkowsky (1984) formulated a general theory for the interaction of a flame with an incident acoustic wave, whose time scale is comparable to the transit time of the flame, $O(d/U_L)$, where d is the flame thickness. McIntosh & Wilce (1990) and McIntosh (1991, 1993) analysed effects of acoustic waves in several distinguished regimes of higher frequencies, while Peters & Ludford (1984) and Keller & Peters (1994) considered the response of a flame to general pressure variations occurring on time scales much longer than $O(d/U_L)$.

The dynamic impact of an acoustic field on the flame was first recognized by Markstein (1953). In a series of experiments, Markstein observed that a flame propagating through a tube developed oscillatory cellular structures, and at the same time significant pressure fluctuations were generated; the frequency of the latter was twice that of the flame cells. He proposed that the spontaneous oscillations arose because of the mutual coupling of two processes: (i) the wrinkling flame modulates the heat release to drive acoustic motions, and (ii) the acoustic pressure distorts the flame through 'acceleration instability' (i.e. the R-T effect). To substantiate this mechanism, Markstein & Squire (1955) isolated (ii) by considering the stability of a flame subjected to an externally prescribed pressure field. They derived a Mathieu-type equation, which corresponds to the one describing the D-L instability but with the time-periodic acoustic acceleration being added to the constant gravity acceleration. The analysis shows that as the forcing exceeds a threshold, which corresponds to an acoustic velocity about three times of the laminar flame speed, a subharmonic parametric resonance leads to a massive instability. Further studies by Searby & Rochwerger (1991) confirmed this instability and in addition found that a moderate level of acoustic forcing stabilizes the flame. Searby (1992) made further experimental study of spontaneous radiation of sound waves by an unforced flame. He found that an initially planar flame develops wrinkles owing to D-L instability, generating a sound wave as a result. For a low-equivalence-ratio propane–air mixture, the flame-induced sound has a stabilizing effect, but at high equivalence ratios, it destabilizes the flame through the subharmonic parametric resonance.

While it has generally been recognized that the mutual interaction between the flame and its acoustic field is of central importance, most theoretical work is concerned with one-way coupling: the impact of an (externally imposed) acoustic pressure on flames. A first-principle description of two-way coupling has to include simultaneously the inverse process, the influence of a flame on the acoustic field, and this has proved to be a major challenge. For a planar flame, Clavin, Pelce & He (1990) showed that the change of the burning rate due to the pressure produces a positive feedback effect on the acoustic field, so that the mutual interaction leads to exponential growth of the sound. Pelce & Rochwerger (1992) investigated acoustic instability of a pre-existing stationary curved flame, where the coupling with the acoustic field is through the heat release associated with the surface-area change. It should be pointed out that this problem is different from that of a developing cellular flame described by Markstein

(1953); the flame-acoustic coupling is linear in the former but nonlinear in the latter.

Based on the work of Matalon & Matkowsky (1982), a general asymptotic theory was presented by Wu *et al.* (2003; referred to as WWMP hereafter) to describe the acoustic-hydrodynamic-flame coupling in the so-called flamelet regime, where the acoustic time scale is assumed to be comparable with the hydrodynamic time scale $O(h^*/U_L)$; here $h^* \gg d$ is the characteristic length of the flow field. This general formulation has been used to provide a unified description of the flame-acoustic coupling mechanisms of Clavin *et al.* (1990) and Pelce & Rochwerger (1992).

In the present paper, the framework of WWMP will be adapted to investigate the development of a stable ducted flame under the influence of a small-amplitude vortical disturbance superimposed on the oncoming mixture. This highly idealized situation allows us to probe into a key aspect in combustion instability: the dynamical interaction between a wrinkling flame and its spontaneously induced acoustic field. The present study is closely related to the observations made by Markstein (1953), which cannot be adequately explained by the one-way interaction theory of Markstein & Squire (1955) and Searby & Rochwerger (1991) because the required large-amplitude external pressure is not usually available in experiments or applications. The key to providing a complete explanation is to consider the two-way coupling and include relevant small-amplitude vortical disturbances in the formulation. Specifically, by analysing the initiation of the subharmonic parametric flame-acoustic resonance by a small-amplitude disturbance on the basis of first principles, a self-consistent mathematical description will be given for the closed-loop coupling mechanism proposed by Markstein (1953). It will be shown that due to this mechanism an otherwise stable flame may become unstable and noisy without any external acoustic forcing, just as was observed in experiments.

The rest of the paper is organized as follows: In §2, we formulate the problem and then synthesize the key results in the hydrodynamic flame theory of Matalon & Matkowsky (1982) to form a 'composite' flame-acoustic interaction theory with $O(\delta)$ accuracy (cf. WWMP), where $\delta = d/h^* \ll O(1)$. The linear 'steady-state' response of the flame to small-amplitude vortical perturbation is calculated in §3. The unsteady heat release of the wrinkled flame drives acoustic oscillations in the duct, the solution of which is obtained in §4. In the case of flame-acoustic resonance, which occurs when the frequency of the vortical disturbance is one half of the frequency of an acoustic mode, the induced acoustic pressure may acquire a sufficiently large amplitude to act back on the flame simultaneously, leading to a fully coupled stage. This scenario is considered in §5 for a flame with a fixed mean position. A system describing the two-way coupling is derived, and the numerical solution shows that the interaction leads to a violent parametric instability of the flame. In §6, we consider a moving flame, for which the resonance is of transient nature, and a slightly modified system is derived. Relevant numerical solutions are presented in §7. Main conclusions are summarized and their implications discussed in §8.

Vortical disturbances in the present flame-acoustic interaction play a role different from that considered in WWMP, where they were assumed to have a frequency equal to that of an acoustic mode. The relevant parameters were such that a marginally unstable D-L mode existed. A resonant triad then emerged consisting of the D-L mode, a weak pre-existing acoustic mode and a vortical disturbance. Owing to the triadic interaction among them, the acoustic pressure amplified considerably but was not able to acquire $O(1)$ amplitude to induce the parametric instability, as it does in the present investigation.

2. Formulation

Consider the combustion of a homogeneous premixed combustible mixture in a duct of width h^* and length $l^* \gg h^*$. The fresh mixture enters the duct at a constant mean velocity U_-^*, but small-amplitude vortical fluctuations are superimposed on the oncoming flow. For simplicity, a one-step irreversible exothermic chemical reaction is assumed. The gaseous mixture consists of a single deficient reactant and an abundant component so that the progress variable of the reaction can be taken to be the mass fraction of the former, Y, and the properties are determined by the latter. The mixture is assumed to satisfy the state equation for a perfect gas.

The fresh mixture has a density $\rho_{-\infty}$ and temperature $\Theta_{-\infty}$. Due to steady heat release, the mean temperature (density) behind the flame increases (decreases) to Θ_∞ (ρ_∞). The non-dimensionalized activation energy, i.e. the Zeldovich number, is defined as

$$\beta = E^*(\Theta_\infty - \Theta_{-\infty})/\mathscr{R}\Theta_\infty^2, \tag{2.1}$$

where E^* is the dimensional activation energy and $\mathscr{R}$ the universal gas constant. The flame propagates into the fresh mixture at a mean speed U_L and has an intrinsic thickness $d = D_{th}^*/U_L$, where D_{th}^* is the thermal diffusivity. For later reference, we define the ratio δ and the Mach number M as

$$\delta = d/h^*, \quad M = U_L/a^*,$$

where $a^* = (\gamma p_{-\infty}/\rho_{-\infty})^{1/2}$ is the speed of sound, with γ denoting the ratio of specific heats. Other relevant parameters are the Prandtl number Pr, Lewis number Le and the normalized gravity force

$$G = gh^*/U_L^2.$$

Let (x, y, z) and t be space and time variables normalized by h^* and h^*/U_L respectively, where x is directed along the streamwise direction and y and z are in the transverse directions. The velocity $\boldsymbol{u} \equiv (u, v, w)$, density ρ and temperature θ are normalized by U_L, $\rho_{-\infty}$ and $\Theta_{-\infty}$ respectively. The non-dimensional pressure p is introduced by writing the dimensional pressure as $(p_{-\infty} + \rho_{-\infty}U_L^2 p)$.

The shear viscosity is assumed to be independent of temperature, and the bulk viscosity is zero. The velocity, pressure, temperature and fuel mass fraction satisfy the non-dimensional Navier–Stokes equations for reactive flows with the reaction rate Ω being described by the Arrhenius law:

$$\Omega = \rho Y \exp\left\{\beta\left(\frac{1}{\Theta_+} - \frac{1}{\theta}\right)\right\}, \tag{2.2}$$

where $\Theta_+ = 1 + q$ is the adiabatic flame temperature.

The mathematical theories for combustion have been developed by assuming a large activation energy and the Lewis number Le close to unity, or more precisely

$$\beta \gg 1, \quad Le = 1 + \beta^{-1}l \quad \text{with} \quad l = O(1). \tag{2.3}$$

The thermal-diffusive theory was first to emerge (see e.g. Clavin 1994 for review), according to which a flame consists of a preheat zone of width $O(d)$, where diffusion balances advection of mass/heat, and a thinner reaction zone with $O(d/\beta)$ width, where species/heat production balances diffusion (Matkowsky & Sivashinsky 1979). This theory ignores the hydrodynamics and gas expansion. Inclusion of these effects led to the hydrodynamic theory for flames (Matalon & Matkowsky 1982; Pelce &

Clavin 1982), which assumes, in addition to $\beta \gg 1$, that

$$\delta = d/h^* \ll 1, \quad M \ll 1. \tag{2.4}$$

The flame-flow problem under consideration comprises of three zones. In addition to the reaction and preheat zones constituting the inner flame structure, a flame generates, through gas expansion, hydrodynamic motions in an $O(h^*)$ region on each side of the flame interface. Interaction of a flame with an externally imposed flow is also 'mediated' by this region. While there is a mean density jump across the flame, the motion on each side is incompressible to leading order. Finally, the flame-hydrodynamic-acoustic interaction theory of WWMP was formulated by introducing outer acoustic zones with a longitudinal length scale $O(h^*/M)$ on either side of the hydrodynamic region. The resulting four regions describe the acoustic, hydrodynamics, heat transfer and chemical reaction and more importantly the intricate interplay among them. They are fully interactive in the sense that the final complete solution relies on the investigation of all these regions. With the preheat and reaction zones being treated analytically, the direct flame-acoustic interaction is between the acoustic and hydrodynamic regions.

2.1. *Summary of the hydrodynamic theory for a flame*

In the hydrodynamic theory of Matalon & Matkowsky (1982) and Pelce & Clavin (1982), the flame front is given by $x = f(y, z, t)$. A coordinate system attached to the front,

$$\xi = x - f(y, z, t), \quad \eta = y, \quad \zeta = z,$$

is introduced, and the velocity $\boldsymbol{u}$ is split as

$$\boldsymbol{u} = u\,\boldsymbol{i} + \boldsymbol{v},$$

where $\boldsymbol{i}$ is the unit vector along the duct.

In the hydrodynamic region, the mean density $R = R_- = 1$ for $\xi < 0$ and $R = R_+ = 1/(1+q)$ for $\xi > 0$. The solution for the flow field and flame front expands as

$$(u,\ \boldsymbol{v},\ p,\ f) = (u_0,\ \boldsymbol{v}_0,\ p_0,\ f_0) + \delta(u_1,\ \boldsymbol{v}_1,\ p_1,\ f_1) + \cdots. \tag{2.5}$$

The leading-order flow field $(u_0,\ \boldsymbol{v}_0,\ p_0)$ satisfies the incompressible Euler equations

$$\left.\begin{aligned} \frac{\partial s_0}{\partial \xi} + \nabla\cdot\boldsymbol{v}_0 &= 0, \\ R\left\{\frac{\partial u_0}{\partial t} + s_0\frac{\partial u_0}{\partial \xi} + \boldsymbol{v}_0\cdot\nabla u_0\right\} &= -\frac{\partial p_0}{\partial \xi} - RG, \\ R\left\{\frac{\partial \boldsymbol{v}_0}{\partial t} + s_0\frac{\partial \boldsymbol{v}_0}{\partial \xi} + \boldsymbol{v}_0\cdot\nabla \boldsymbol{v}_0\right\} &= -\nabla p_0 + \nabla f_0\,\frac{\partial p_0}{\partial \xi} \end{aligned}\right\} \tag{2.6}$$

on either side of the flame, where

$$s_0 = u_0 - f_{0,t} - \boldsymbol{v}_0\cdot\nabla f_0. \tag{2.7}$$

Equations (2.6) are coupled with the front equation

$$f_{0,t} = u_0(0^-, \eta, \zeta, t) - \boldsymbol{v}_0(0^-, \eta, \zeta, t)\cdot\nabla f_0 - [1 + (\nabla f_0)^2]^{1/2}. \tag{2.8}$$

The flow fields separated by the flame interface are linked through the jumps

$$[[u_0]] = q/m_0, \quad [[\boldsymbol{v}_0]] = -q\nabla f_0/m_0, \quad [[p_0]] = -q, \tag{2.9}$$

where

$$m_0 = [1 + (\nabla f_0)^2]^{1/2}.$$

The leading-order system (2.6)–(2.9) suffices for many purposes (e.g. stability calculations) but is ill posed as an initial-value problem because D-L instability modes amplify at rates proportional to their wavelengths. A regularized problem may be derived by including the $O(\delta)$ corrections in the hydrodynamic region and in the jump conditions across the preheat zone. The jumps at this order were obtained by Matalon & Matkowsky (1982) for the nonlinear case without gravity, and their analysis was repeated in WWMP with a minor extension of including the gravity effect. A further modification is required due to the spontaneous acoustic field of the flame, which generates an $O(\beta M)$ enthalpy fluctuation (Clavin *et al.* 1990). The enthalpy impinges on the flame to drive an $O(\beta M)$ velocity fluctuation. This part of motion was considered in WWMP and can be conveniently grouped with the $O(\delta)$ terms by tacitly assuming that

$$\delta_M \equiv \beta M/\delta = O(1). \tag{2.10}$$

Then the $O(\delta)$ jump in u_1 obtained by Matalon & Matkowsky (1982) is modified to

$$\begin{aligned}[[u_1]] = &-\frac{q}{m_0^3}\nabla f_0\cdot\nabla f_1 - \frac{lq\mathrm{D}(q)}{2m_0^2}\left\{\nabla^2 f_0 + m_0\nabla\cdot\boldsymbol{v}_0 + \frac{\mathrm{D}m_0}{\mathrm{D}t}\right\} + \frac{q}{m_0}\left(\frac{1}{2}\delta_M\tilde{h}\right),\\ &+\chi m_0\left\{\left[\left[\frac{\partial\,\boldsymbol{v}_0}{\partial\xi}\cdot\nabla f_0\right]\right] - q\nabla\cdot\left(\frac{\nabla f_0}{m_0}\right) - \frac{q}{m_0^3}\nabla^2 f_0 + \frac{2q}{m_0^4}\nabla m_0\cdot\nabla f_0\right\},\end{aligned} \tag{2.11}$$

where

$$\tilde{h} = (\gamma - 1)p_a(0, t) \tag{2.12}$$

with $p_a(0, t)$ being the scaled acoustic pressure at the flame front (see next subsection). The leading-order jump $[[u_0]]$ in (2.9) and $[[u_1]]$ can be combined to give $[[u]]$ to $O(\delta)$ accuracy. Note that the $-(q/m_0^3)\nabla f_0\cdot\nabla f_1$ in (2.11) is simply the second term in the expansion of q/m for $f = f_0 + \delta f_1 + O(\delta^2)$. This term can be absorbed into q/m, if we collect the first two terms in the expansions for f and u as a 'synthesized' approximation, which is actually more convenient to use. In general, by such synthesis it follows that $[[u]]$, to $O(\delta)$ accuracy, can be written as (cf. Matalon & Matkowsky 1982)

$$[[u]] = \frac{q}{m}\left(1 + \frac{1}{2}\beta M\tilde{h}\right) + \delta\left\{-\frac{ql\mathscr{D}}{2m^2}\Gamma + \frac{\chi q}{(1+q)m^2}\boldsymbol{\Gamma}_{\boldsymbol{v}}\cdot\nabla f\right\}, \tag{2.13}$$

where

$$\Gamma = \nabla^2 f + m\nabla\cdot\boldsymbol{v}^- + \frac{\mathrm{D}m}{\mathrm{D}t}, \qquad \boldsymbol{\Gamma}_{\boldsymbol{v}} = \frac{\tilde{\mathrm{D}}\,\boldsymbol{v}^-}{\tilde{\mathrm{D}}t} + \nabla f\frac{\tilde{\mathrm{D}}u^-}{\tilde{\mathrm{D}}t} + \frac{1}{m}\frac{\tilde{\mathrm{D}}}{\tilde{\mathrm{D}}t}\nabla f + G\nabla f.$$

Similarly, the transverse velocity and pressure jumps can be written as

$$[[\boldsymbol{v}]] = -[[u]]\nabla f + \delta\frac{\chi q}{1+q}\boldsymbol{\Gamma}_{\boldsymbol{v}}, \tag{2.14}$$

$$\begin{aligned}[[p]] = -2m[[u]] + \delta\Bigg\{q\nabla\cdot\left(\frac{\nabla f}{m}\right) + \left(m\frac{\tilde{\mathrm{D}}u^-}{\mathrm{D}t} + \frac{1}{m}\frac{\tilde{\mathrm{D}}m}{\mathrm{D}t} + mG\right)\ln(1+q)\\ + \frac{q(Pr+\chi)}{(1+q)m}\boldsymbol{\Gamma}_{\boldsymbol{v}}\cdot\nabla f\Bigg\},\end{aligned} \tag{2.15}$$

where u^- and $\boldsymbol{v}^-$ and their derivatives are evaluated at the flame front; $\xi = 0^-$; and

$$\mathscr{D}(q) = \int_0^\infty \ln(1 + q\,\mathrm{e}^{-x})\,\mathrm{d}x, \quad \chi = Pr + \frac{1+q}{q}\ln(1+q),$$
$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{v}^- \cdot \nabla, \qquad \frac{\tilde{\mathrm{D}}}{\tilde{\mathrm{D}}t} = \frac{\mathrm{D}}{\mathrm{D}t} + \frac{\nabla f \cdot \nabla}{m},$$

as in Matalon & Matkowsky (1982). The relation (2.14) can alternatively be expressed as

$$[[\boldsymbol{v}]] = -[[u]]\nabla f + \delta\left\{Pr\, m\left[\left[\frac{\partial}{\partial \xi}(\boldsymbol{v} + u\nabla f)\right]\right] + q\,Pr\frac{\nabla m}{m} + \ln(1+q)\boldsymbol{\Gamma}_v\right\}. \tag{2.16}$$

The function f satisfies the equation (cf. Matalon & Matkowsky 1982)

$$f_t = u^- - \boldsymbol{v}^- \cdot \nabla f - m\left(1 + \frac{1}{2}\beta M \tilde{h}\right) + \delta M_a\left\{\nabla^2 f + m\nabla \cdot \boldsymbol{v}^- + \frac{\mathrm{D}m}{\mathrm{D}t}\right\}, \tag{2.17}$$

where

$$M_a = \frac{1+q}{q}\ln(1+q) + \tfrac{1}{2} l \mathscr{D}.$$

The linearized version of (2.13)–(2.17) was given by Pelce & Clavin (1982). As is well known, the $O(\delta)$ terms account for the effects of thermal diffusion and flame curvature. They stabilize D-L modes with wavelengths shorter than a cutoff length, provided that the Lewis number is sufficiently close to unity, and consequently the initial-value problem becomes well posed. The D-L instability may be suppressed completely across all wavelengths, and a planar flame is rendered intrinsically stable when the laminar flame speed U_L and/or the gas expansion parameter q are sufficiently small.

2.2. *A general composite theory of flame-acoustic interaction*

2.2.1. *Acoustic zone*

The acoustic-flame coupling theory of WWMP results from introducing an outer acoustic region. The appropriate variable describing the acoustic motion in this region is

$$\tilde{\xi} = M\xi. \tag{2.18}$$

The motion is a longitudinal oscillation about the uniform mean background, and the solution can be written as

$$(u, \rho, \theta) = (U_\pm, R_\pm, \Theta_\pm) + (u_a(\tilde{\xi}, t), M\rho_a(\tilde{\xi}, t), M\theta_a(\tilde{\xi}, t)), \quad p = M^{-1} p_a(\tilde{\xi}, t), \tag{2.19}$$

where $U_\pm$ are the mean velocities of the burned and the fresh mixture respectively, with $U_+ - U_- = q$. Strictly speaking, the acoustic field $(u_a, p_a, \theta_a, \rho_a)$ should formally be expanded as an asymptotic series in terms of the small parameters δ and βM, since the source consists of terms at these orders (see (2.22)). However, it is convenient to take a composite approach, which keeps all these terms together with the leading-order term.

The unsteady field is governed by the linearized acoustic equations. Elimination of θ_a and ρ_a among them yields the wave equations for velocity u_a and pressure p_a,

$$R\frac{\partial^2 u_a}{\partial t^2} - \frac{\partial^2 u_a}{\partial \tilde{\xi}^2} = 0, \qquad R\frac{\partial u_a}{\partial t} = -\frac{\partial p_a}{\partial \tilde{\xi}}. \tag{2.20}$$

As $\tilde{\xi} \to \pm 0$,

$$u_a \to u_a(0^\pm, t) + \cdots, \quad p_a \to p_a(0, t) + p_{a,\tilde{\xi}}(0^\pm, t)\tilde{\xi} + \cdots.$$

As will be shown in the next subsection, the acoustic pressure is continuous across the flame, but the flame induces a jump in u_a, i.e.

$$[p_a] = 0\ , \tag{2.21}$$

$$\begin{aligned}[u_a] \equiv \mathscr{J} = q\left\{\overline{(1+(\nabla F)^2)^{1/2}} - 1\right\} - \delta\frac{ql\mathscr{D}}{2}\frac{\partial}{\partial t}\overline{(1+(\nabla F)^2)^{1/2}} \\ + \beta M q\left(\frac{1}{2}\tilde{h}\right)\overline{(1+(\nabla F)^2)^{1/2}},\end{aligned} \tag{2.22}$$

where $\overline{\phi}$ denotes the space average of ϕ in the (η, ζ)-plane and F is defined in (2.34) below. The jump $[u_a]$ acts as an acoustic source, through which the flame may excite acoustic modes of the duct. The result (2.22) provides an explicit relation between the source of sound and the property of the flame. As expected on physical ground, the primary source corresponds to alteration of the flame surface area. However, there is also an $O(\delta)$ contribution by the instantaneous rate of change of the surface area. This secondary source, which vanishes for unity Lewis number, is associated with the thermal-diffusive structure of the flame and acts to moderate the main contribution in (2.22). The $O(\beta M)$ term is due to the effect of the acoustic pressure on the burning rate, and it extends the result of Clavin *et al.* (1990) for a planar flame to a curved flame in the corrugated flamelet regime. Unlike other terms in (2.22), it represents an amplification rather than a generation mechanism of sound waves, since the forcing $\tilde{h}$ is the acoustic pressure (see (2.12)).

At the two ends of the duct, the following conditions may be imposed:

$$u_a = 0 \quad \text{at} \quad \tilde{\xi} = -\sigma L, \qquad p_a = 0 \quad \text{at} \quad \tilde{\xi} = (1-\sigma)L, \tag{2.23}$$

where $L = Ml^*/h^*$ and σ is a parameter characterizing the mean position of the flame front.

For an arbitrary forcing $\mathscr{J}(t)$, a formal general solution for the acoustic field can be constructed by introducing the *causal* Green's function $\boldsymbol{G}(\tilde{\xi}, t|\tau)$, defined as

$$\left.\begin{aligned} R_\pm\frac{\partial^2 \boldsymbol{G}}{\partial t^2} - \frac{\partial^2 \boldsymbol{G}}{\partial \tilde{\xi}^2} = 0, \\ \boldsymbol{G} = 0 \text{ at } \tilde{\xi} = -\sigma L, \quad \partial\boldsymbol{G}/\partial\tilde{\xi} = 0 \text{ at } \tilde{\xi} = (1-\sigma)L, \\ [\boldsymbol{G}] = \delta(t-\tau), \quad \left[\frac{\partial \boldsymbol{G}}{\partial\tilde{\xi}}\right] = 0 \text{ at } \tilde{\xi} = 0, \\ \boldsymbol{G} = \partial\boldsymbol{G}/\partial t = 0 \text{ for } t < \tau. \end{aligned}\right\} \tag{2.24}$$

Obviously, $\boldsymbol{G}$ depends on t and τ through the combination $(t-\tau)$, i.e. $\boldsymbol{G} = \boldsymbol{G}(\tilde{\xi}\,; t-\tau)$. In terms of $\boldsymbol{G}$, the general solution for u_a can be written as

$$u_a(\tilde{\xi}, t) = \int_0^t \boldsymbol{G}(\tilde{\xi}\,; t-\tau)\mathscr{J}(\tau)\,\mathrm{d}\tau,$$

where we have used the fact that $\mathscr{J}(t) = 0$ for $t < 0$.

The system (2.24) can be solved by taking Fourier transform with respect to $(t-\tau)$, to obtain the Fourier transform of $\boldsymbol{G}$, defined as

$$\widehat{\boldsymbol{G}}(\tilde{\xi}\,;\omega)=\int_{-\infty}^{\infty}\boldsymbol{G}(\tilde{\xi}\,;\,t-\tau)\,\mathrm{e}^{-\mathrm{i}\,\omega(t-\tau)}\,\mathrm{d}(t-\tau).$$

It is found that

$$\widehat{\boldsymbol{G}}(\tilde{\xi}\,;\omega)=\begin{cases}-\dfrac{\left(\dfrac{R_+}{R_-}\right)^{1/2}\tan\left(R_+^{1/2}(1-\sigma)\omega L\right)}{\Delta_s(\omega;\sigma)\cos\left(R_-^{1/2}\sigma\omega L\right)}\sin\left[R_-^{1/2}(\tilde{\xi}+\sigma\omega L)\right], & \tilde{\xi}<0,\\[2ex] -\dfrac{\cos\left[R_+^{1/2}(\tilde{\xi}-(1-\sigma)\omega L)\right]}{\Delta_s(\omega;\sigma)\cos\left[R_+^{1/2}(1-\sigma)\omega L\right]}, & \tilde{\xi}>0,\end{cases}\qquad(2.25)$$

where

$$\Delta_s(\omega;\sigma)=\left(\frac{R_+}{R_-}\right)^{1/2}\tan\left(R_-^{1/2}\sigma\omega L\right)\tan\left(R_+^{1/2}(1-\sigma)\omega L\right)-1.\qquad(2.26)$$

The characteristic frequencies of acoustic modes of the duct are given by the eigenrelation

$$\Delta_s(\omega_k\,;\sigma)=0,\qquad(2.27)$$

which has countable number of roots, ω_k $(k=1,2,\ldots)$ say. The inverse of (2.25) gives

$$\boldsymbol{G}(\tilde{\xi}\,;t-\tau)=\frac{1}{2\pi}\int_{-\infty-\mathrm{i}\,0}^{\infty-\mathrm{i}\,0}\widehat{\boldsymbol{G}}(\tilde{\xi}\,;\omega)\,\mathrm{e}^{\mathrm{i}\,\omega(t-\tau)}\,\mathrm{d}\omega,\qquad(2.28)$$

where '$-\,\mathrm{i}\,0$' indicates that the contour in the complex ω-plane is taken to be a line below the real poles (zeros) of $\widehat{\boldsymbol{G}}$ (Δ_s) in order to ensure causality. Note that $\widehat{\boldsymbol{G}}$ remains bounded so that the entire integrand decays exponentially as $\mathrm{Im}(\omega)>0$. By closing the contour in the upper half-plane and using the residual theorem, the inversion can be evaluated analytically to obtain

$$\boldsymbol{G}(\tilde{\xi}\,;t-\tau)=\begin{cases}-\displaystyle\sum_k\frac{i\left(\frac{R_+}{R_-}\right)^{1/2}\tan\left(R_+^{1/2}(1-\sigma)\omega_k L\right)}{\Delta_s'(\omega_k\,;\sigma)\cos\left(R_-^{1/2}\sigma\omega_k L\right)}\sin\left[R_-^{1/2}(\tilde{\xi}+\sigma\omega_k L)\right]\mathrm{e}^{\mathrm{i}\,\omega_k(t-\tau)}, & \tilde{\xi}<0,\\[2ex] -\displaystyle\sum_k\frac{\mathrm{i}\cos\left[R_+^{1/2}(\tilde{\xi}-(1-\sigma)\omega_k L)\right]}{\Delta_s'(\omega_k\,;\sigma)\cos\left[R_+^{1/2}(1-\sigma)\omega_k L\right]}\mathrm{e}^{\mathrm{i}\,\omega_k(t-\tau)}, & \tilde{\xi}>0.\end{cases}\qquad(2.29)$$

Since the analysis of acoustic-flame interaction requires the acoustic signature at the flame front $\tilde{\xi}=0^{\pm}$ only, it suffices to consider $\boldsymbol{G}(0^{\pm};t-\tau)$, which resumes a simpler expression

$$\boldsymbol{G}(0^{\pm};t-\tau)=-\delta(t-\tau)h(-0^{\pm})-\sum_k\frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\,\omega_k(t-\tau)}}{\Delta_s'(\omega_k\,;\sigma)}\,\mathrm{d}\omega,\qquad(2.30)$$

where $h(-0^-)=1$ and $h(-0^+)=0$, and the summation is over all roots of the acoustic dispersion relation (2.27). Use of (2.30) allows for calculation of the local acoustic velocity

$$u_a(0^{\pm},t) = -\mathscr{J}(t)h(-0^{\pm}) - \sum_k \frac{\mathrm{i}}{\Delta_s'(\omega_k;\sigma)} \int_0^t \mathrm{e}^{\mathrm{i}\,\omega_k(t-\tau)}\,\mathscr{J}(\tau)\,\mathrm{d}\tau. \tag{2.31}$$

For the purpose of comparison, we note that the Green function for an unbounded domain is

$$\boldsymbol{G}(\tilde{\xi};t-\tau) = \pm R_{\mp}^{1/2} \big/ \left(R_+^{1/2} + R_-^{1/2}\right) \delta\left(t-\tau \mp R_{\pm}^{1/2}\tilde{\xi}\right). \tag{2.32}$$

It then follows that

$$u_a(\tilde{\xi},t) = \pm R_{\mp}^{1/2} \big/ \left(R_+^{1/2} + R_-^{1/2}\right) \mathscr{J}\left(t \mp R_{\pm}^{1/2}\tilde{\xi}\right). \tag{2.33}$$

The result (2.31) indicates that a flame in a bounded domain may be in resonance with natural acoustic modes of the chamber (duct), for which case $\mathscr{J}$ would behave approximately as $\mathrm{e}^{\mathrm{i}\omega_k t}$ so that the second term in (2.31) would be very large. Such a resonance is of course not possible for a flame in open space, and as a result the back effect of spontaneous sound waves on the flame, though also present, is likely to be weak.

2.2.2. *Hydrodynamic zone*

The formulation of WWMP can be extended to construct a composite theory with $O(\delta)$ accuracy by retaining the $O(\delta)$ viscous effect in the hydrodynamic zone and by using the synthesized $O(\delta)$ accurate jumps (2.13)–(2.15).

In the hydrodynamic zone, the velocity and pressure are decomposed as

$$\left.\begin{aligned} &u = U_{\pm} + u_a(0^{\pm},t) + U, \qquad f = F_a(t) + F, \\ &p = M^{-1}p_a(0,t) + P_{\pm} + \left(p_{a,\tilde{\xi}}(0^{\pm},t) - RG\right)(\xi + F) + P, \end{aligned}\right\} \tag{2.34}$$

where $P_{\pm}$ is the mean pressure (with $P_+ - P_- = -q$) and

$$F_a'(t) = U_- - 1 + u_a(0^-,t) - \frac{1}{2}\beta M\tilde{h}. \tag{2.35}$$

The flame front equation (2.17), to $O(\delta)$ accuracy, may be written as

$$F_t + \boldsymbol{V}^- \cdot \nabla F = U^- - (m-1)\left(1 + \frac{1}{2}\beta M\tilde{h}\right) + \delta M_a\left\{\nabla^2 F + m\nabla\cdot\boldsymbol{V}^- + \frac{\mathrm{D}m}{\mathrm{D}t}\right\}, \tag{2.36}$$

where we have put $\boldsymbol{v} = \boldsymbol{V}$.

The composite equations governing the hydrodynamics therefore read (cf. WWMP)

$$\left.\begin{aligned} \frac{\partial U}{\partial \xi} + \nabla\cdot\boldsymbol{V} &= \frac{\partial \boldsymbol{V}}{\partial \xi}\cdot\nabla F, \\ R\left(\frac{\partial \tilde{U}}{\partial t} + S\frac{\partial U}{\partial \xi} + \boldsymbol{V}\cdot\nabla U\right) + (1 + R\mathscr{J}h(\xi))\frac{\partial U}{\partial \xi} &= -\frac{\partial P}{\partial \xi} + \delta Pr\Delta U, \\ R\left(\frac{\partial \boldsymbol{V}}{\partial t} + S\frac{\partial \boldsymbol{V}}{\partial \xi} + \boldsymbol{V}\cdot\nabla\boldsymbol{V}\right) + (1 + R\mathscr{J}h(\xi))\frac{\partial \boldsymbol{V}}{\partial \xi} &= -\nabla P + \nabla F\frac{\partial P}{\partial \xi} + \delta Pr\Delta\boldsymbol{V}, \end{aligned}\right\} \tag{2.37}$$

where $h(\xi)$ is the Heaviside step function and

$$\mathscr{J} = [u_a], \quad S = U - F_t - \boldsymbol{V}\cdot\nabla F.$$

The Laplace operator Δ is defined by

$$\triangle = (1+(\nabla F)^2)\frac{\partial^2}{\partial \xi^2} + \nabla^2 - \nabla^2 F\frac{\partial}{\partial \xi} - 2\frac{\partial}{\partial \xi}(\nabla F\cdot\nabla),$$

with the operators ∇ and ∇^2 being defined with respect to η and ζ. The velocity and pressure satisfy the upstream condition,

$$(U,\ \boldsymbol{V}) \to (U_e, \boldsymbol{V}_e)\ , \quad P_\xi \to 0 \quad \text{as} \quad \xi \to -\infty, \tag{2.38}$$

in order to match with the oncoming vortical disturbance $(U_e, \boldsymbol{V}_e)$.

By the same procedure as in § 3 of WWMP, it can be shown that

$$\mathscr{J} = [u_a] = -\overline{[[\boldsymbol{V}]]\cdot\nabla F} + \overline{[[u]]-q}, \tag{2.39}$$

$$[[U]] = \left([[u]] - \overline{[[u]]}\right) + \overline{[[\boldsymbol{V}]]\cdot\nabla F}. \tag{2.40}$$

Inserting (2.13) and (2.14) into the above equations, we obtain (2.22) and

$$\begin{aligned}[][[U]] = {} & q\left\{(1+(\nabla F)^2)^{-1/2} - \overline{(1+(\nabla F)^2)^{1/2}}\right\}\left(1+\frac{1}{2}\beta M\tilde{h}\right)\\ & + \delta\left\{-\frac{ql\mathscr{D}}{2m^2}\left[\nabla^2 F + \nabla\cdot(m\,\boldsymbol{V}^-) + \frac{\partial m}{\partial t} - m^2\frac{\partial \overline{m}}{\partial t}\right]\right.\\ & \left. + \frac{\chi q}{(1+q)m^2}\left[\frac{\tilde{\mathrm{D}}\,\boldsymbol{V}^-}{\tilde{\mathrm{D}}t} + \nabla F\frac{\tilde{\mathrm{D}}U^-}{\tilde{\mathrm{D}}t} + \frac{1}{m}\frac{\tilde{\mathrm{D}}}{\tilde{\mathrm{D}}t}\nabla F + (G + u_{a,t}(0^-,t))\nabla F\right]\cdot\nabla F\right\}.\end{aligned} \tag{2.41}$$

The transverse velocity jump (2.14) may be rewritten as

$$[[\boldsymbol{V}]] = -[[u]]\nabla F + \frac{\delta\chi q}{(1+q)}\left[\frac{\tilde{\mathrm{D}}\,\boldsymbol{V}^-}{\tilde{\mathrm{D}}t} + \nabla F\frac{\tilde{\mathrm{D}}U^-}{\tilde{\mathrm{D}}t} + \frac{1}{m}\frac{\tilde{\mathrm{D}}}{\tilde{\mathrm{D}}t}\nabla F + \left(G + u_{a,t}(0^-,t)\right)\nabla F\right] \tag{2.42}$$

or, if (2.16) is preferred,

$$\begin{aligned}[][[\boldsymbol{V}]] = {} & -[[u]]\,\nabla F + \delta\left\{Pr\,m\left[\left[\frac{\partial}{\partial \xi}(\boldsymbol{V} + U\nabla f)\right]\right] + q\,Pr\frac{\nabla m}{m}\right.\\ & \left. + \ln(1+q)\left[\frac{\tilde{\mathrm{D}}\,\boldsymbol{V}^-}{\tilde{\mathrm{D}}t} + \nabla F\,\frac{\tilde{\mathrm{D}}U^-}{\tilde{\mathrm{D}}t} + \frac{1}{m}\frac{\tilde{\mathrm{D}}}{\tilde{\mathrm{D}}t}\nabla F + \left(G + u_{a,t}(0^-,t)\right)\nabla f\right]\right\}.\end{aligned} \tag{2.43}$$

The pressure jump (2.15) becomes

$$\begin{aligned}[][[P]] = {} & [(R_+ - R_-)G - \Delta p_{a,\xi}]F - 2m[[u]]\\ & + \delta\left\{\frac{q}{m}\nabla^2 F + \left[m\frac{\tilde{\mathrm{D}}U^-}{\tilde{\mathrm{D}}t} + \frac{1}{m}\frac{\tilde{\mathrm{D}}m}{\tilde{\mathrm{D}}t} + m(G + u_{a,t}(0^-,t))\right]\ln(1+q) - \frac{q}{m^2}\nabla F\cdot\nabla m\right.\\ & \left. + \frac{q(Pr+\chi)}{(1+q)m}\left[\frac{\tilde{\mathrm{D}}\,\boldsymbol{V}^-}{\tilde{\mathrm{D}}t} + \nabla F\frac{\tilde{\mathrm{D}}U^-}{\tilde{\mathrm{D}}t} + \frac{1}{m}\frac{\tilde{\mathrm{D}}}{\tilde{\mathrm{D}}t}\nabla F + \left(G + u_{a,t}(0^-,t)\right)\nabla F\right]\cdot\nabla F\right\},\end{aligned} \tag{2.44}$$

where we have put

$$\Delta p_{a,\xi}(t) = p_{a,\xi}(0^+, t) - p_{a,\xi}(0^-, t) \equiv -\{R_+ u_{a,t}(0^+, t) - R_- u_{a,t}(0^-, t)\}. \qquad (2.45)$$

This term represents the unsteady R-T effect, arising from the acoustic acceleration $u_{a,t}$ (or local pressure gradient), since $u_{a,t}$ plays the same role as the gravity G. Neglecting this effect in the G-equation approach constitutes its severest limitation.

The result (2.22) indicates that an unsteady curved flame must generate an acoustic field. Meanwhile, relations (2.44) and (2.45) show the emitted sound waves act on the flame through the R-T effect. For freely propagating flames, barring the direct effect on the burning, this is the sole mechanism by which the acoustic field affects the flame because the acoustic velocity causes merely a 'rigid' vibration of the flame without inducing any distortion. For anchored flames, acoustic pressure and velocity are both important.

The great majority of theoretical and computational work on flame dynamics and combustion, however, excludes *a priori* the acoustic field at the outset. This amounts to an *ad hoc* approximation, since an acoustic field is, as (2.22) and (2.31) (or (2.33)) imply, an intrinsic and inseparable part of an unsteady flame. Mathematical inconsistency could arise as analytical or numerical solutions obtained in this manner may contradict the approximation made. These solutions ought to be scrutinized closely before being accepted. For example, the acoustic field of a time-dependent solution has to be estimated posteriorly (e.g. by solving the acoustic equation subject to (2.22)), and only when it is found to be weak may the solution qualify for a mathematically acceptable approximation; otherwise the solution is most likely to be a spurious one. A steady solution is deemed to be mathematically acceptable, but its physically realizability depends on its stability. To formulate an appropriate stability theory, one must recognize that a small perturbation to a general curved flame would necessarily cause an acoustic perturbation of the same order of magnitude. The latter must therefore be included in the stability analysis, and its presence may drastically alter the stability property (cf. Pelce & Rochwerger 1992).

The composite theory of flame-acoustic interaction is valid under the assumptions (2.3), (2.4) and (2.10). For the general case of a strongly wrinkling flame (i.e. $\nabla F \sim O(1)$) and an order-one flow field, the error in the theory is $O(\delta^2 + M)$, arising from neglecting $O(\delta^2)$ effects in the hydrodynamic region and $O(M)$ convection terms in the acoustic equations. In the rest of the paper the general fully nonlinear theory will be specialized to small perturbations, where F and (U, V, W, P) have an amplitude $\epsilon \ll 1$. Analytical progress becomes possible, since the hydrodynamic equations and the jumps can then be linearized. This approximation results in an additional error $O(\epsilon^2)$. On the other hand, we shall retain the $O(\delta\epsilon)$ linear corrections in order to render the initial-value problem well posed. For the neglected $O(\epsilon^2)$ nonlinear terms to be smaller, it would formally be required that $\epsilon \ll \delta$. This condition is, however, unlikely to be met in practice, since δ is usually quite small ($O(10^{-3})$). As a result, the $O(\epsilon)$ *relative* error due to neglecting the hydrodynamic nonlinearity may be greater than $O(\delta)$. A regular perturbation procedure can in principle be adopted to eliminate the $O(\epsilon)$ relative error by including the effect of $O(\epsilon^2)$ nonlinear terms. Refinement along this line requires a lengthy calculation and will be pursued here. As we shall argue, an analysis based on linearized hydrodynamics should be able to capture qualitative behaviours of the flame-acoustic interaction, provided that ϵ is reasonably small. It is worth stressing that despite the presence of the $O(\epsilon)$ error, the validity of the ensuing analysis is ensured by conditions (2.3), (2.4) and (2.10).

Finally, it is worth noting that after subtracting out the local acoustic signature $u_a(0^{\pm}, t)$, the longitudinal velocity jump $[[U]]$ of the pure hydrodynamic field, (2.41), consists of two transversely averaged terms. Taking into account the leading-order one, we repeated the weak-heat-release analysis ($q \ll O(1)$) of Sivashinsky & Clavin (1987) to extend the M-S equation to $O(q)$ and found that the modified jump gives rise to the transversely averaged term that was found to be missing by Joulin & Cambray (1992). The latter authors retrieved this term on physical ground. We now note that it can actually be derived systematically.

3. Linear response and flame wrinkling

A vortical disturbance in the form of convected gusts is assumed to be superimposed on the oncoming fresh mixture. In general, the disturbance has a continuous spectrum and may be represented by an integral over all Fourier components (Searby & Clavin 1986; Aldredge & Williams 1991):

$$(u, \boldsymbol{v}) = \epsilon \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (C_-(\boldsymbol{k}^{\dagger}, \omega), \boldsymbol{D}_-(\boldsymbol{k}^{\dagger}, \omega))\, \mathrm{e}^{\mathrm{i}(k_1\xi + k_2\eta + k_3\zeta - \omega t)}\, \mathrm{d}\boldsymbol{k}^{\dagger}\, \mathrm{d}\omega, \tag{3.1}$$

where ϵ measures the magnitude; $\boldsymbol{k}^{\dagger} = (k_2, k_3)$ is the transverse wavenumber vector; and $k_1 = \omega R_-$ to leading order, since the disturbance is advected passively by the uniform background flow. Functions $C_-(\boldsymbol{k}^{\dagger}, \omega)$ and $\boldsymbol{D}_-(\boldsymbol{k}^{\dagger}, \omega)$ characterize the spectral property. For a small-amplitude disturbance ($\epsilon \ll O(1)$) of interest in this paper, each component can be treated independently, and so it suffices to focus on a single Fourier mode. The oncoming flow field then takes the form

$$(u, \boldsymbol{v}) = \epsilon (C_-, \boldsymbol{D}_-)\, \mathrm{e}^{\mathrm{i}(k_1\xi + k_2\eta + k_3\zeta - \omega t)} + \text{c.c.} \tag{3.2}$$

Note that pressure fluctuation is absent.

The oncoming disturbance induces a hydrodynamical field. For $\epsilon \ll O(1)$, the governing equations (2.37) can be linearized. Hereafter throughout the paper, a planar flame is assumed to be established in the stable regime of the parameter space. Then a steady-state response can be reached, for which the solution takes the form

$$(U, \boldsymbol{V}, P, F) = \epsilon (\bar{U}, \bar{\boldsymbol{V}}, \bar{P}, \bar{F})\, \mathrm{e}^{\mathrm{i}(k_2\eta + k_3\zeta - \omega t)} + \text{c.c.}, \tag{3.3}$$

where $\bar{F}$ is independent of ξ. After substituting it into linearized equations (2.37), it is easily found that

$$\bar{P} = P_{\pm}\, \mathrm{e}^{\mp k\xi}, \quad \bar{U} = \frac{\mp k P_{\pm}\, \mathrm{e}^{\mp k\xi}}{\mathrm{i}\, R_{\pm}\omega \pm k} + C_{\pm}\, \mathrm{e}^{\mathrm{i}\mathscr{S}_{\pm}\xi}, \quad \bar{\boldsymbol{V}} = \frac{\mathrm{i}\, \boldsymbol{k}^{\dagger}\, P_{\pm}\, \mathrm{e}^{\mp k\xi}}{\mathrm{i}\, R_{\pm}\omega \pm k} + \boldsymbol{D}_{\pm}\, \mathrm{e}^{\mathrm{i}\mathscr{S}_{\pm}\xi}, \tag{3.4}$$

where $k = (k_2^2 + k_3^2)^{1/2}$ and

$$\mathscr{S}_{\pm} = \{-\mathrm{i} - \mathrm{i}(1 - 4(\mathrm{i}\,\omega R_{\pm}\delta Pr - k^2\delta^2 Pr^2))^{1/2}\}/(2\delta Pr) = \omega R_{\pm} + \mathrm{i}\, Pr(k^2 + \omega^2 R_{\pm}^2)\delta + O(\delta^2).$$

The constants $C_{\pm}$ and $\boldsymbol{D}_{\pm}$ are related by

$$\mathscr{S}_{\pm} C_{\pm} + \boldsymbol{k}^{\dagger} \cdot \boldsymbol{D}_{\pm} = 0, \tag{3.5}$$

so as to satisfy the continuity equation. Inserting the solution into the linearized jump conditions (2.41)–(2.44) and front equation (2.36), we obtain

$$\left.\begin{aligned} P_+ - P_- &= (R_+ - R_-)G\bar{F} + \delta J_p, \\ \frac{-kP_+}{\mathrm{i}\,R_+\omega + k} + C_+ &= \frac{kP_-}{\mathrm{i}\,R_-\omega - k} + C_- + \delta J_u, \\ \frac{\mathrm{i}\,\boldsymbol{k}^\dagger\,P_+}{\mathrm{i}\,R_+\omega + k} + \boldsymbol{D}_+ &= \frac{\mathrm{i}\,\boldsymbol{k}^\dagger\,P_-}{\mathrm{i}\,R_-\omega - k} + \boldsymbol{D}_- - \mathrm{i}\,q\,\boldsymbol{k}^\dagger\,\bar{F} + \delta \boldsymbol{J}_{\boldsymbol{v}}, \\ -\mathrm{i}\,\omega\bar{F} &= \frac{kP_-}{\mathrm{i}\,R_-\omega - k} + C_- + \delta J_F, \end{aligned}\right\} \tag{3.6}$$

where

$$\begin{aligned} J_p &= lq\mathscr{D}\left[-k^2\bar{F} - \frac{k^2P_-}{\mathrm{i}\,R_-\omega - k} + \mathrm{i}\,\boldsymbol{k}^\dagger\cdot\boldsymbol{D}_-\right] - qk^2\bar{F} - \mathrm{i}\,\omega\ln(1+q)\left[\frac{kP_-}{\mathrm{i}\,R_-\omega - k} + C_-\right], \\ J_u &= -\frac{1}{2}lq\mathscr{D}\left[-k^2\bar{F} - \frac{k^2P_-}{\mathrm{i}\,R_-\omega - k} + \mathrm{i}\,\boldsymbol{k}^\dagger\cdot\boldsymbol{D}_-\right], \\ \boldsymbol{J}_{\boldsymbol{v}} &= \frac{q\chi}{1+q}\left\{(-\mathrm{i}\,\omega)\frac{\mathrm{i}\,\boldsymbol{k}^\dagger\,P_-}{\mathrm{i}\,R_-\omega - k} - \mathrm{i}\,\omega\boldsymbol{D}_- + (-\mathrm{i}\,\omega)\,\mathrm{i}\,\boldsymbol{k}^\dagger\,\bar{F} + G\,\mathrm{i}\,\boldsymbol{k}^\dagger\,\bar{F}\right\}, \\ J_F &= \left(\frac{1+q}{q}\ln(1+q) + \frac{1}{2}l\mathscr{D}\right)\left[-k^2\bar{F} - \frac{k^2P_-}{\mathrm{i}\,R_-\omega - k} + \mathrm{i}\,\boldsymbol{k}^\dagger\cdot\boldsymbol{D}_-\right]. \end{aligned} \tag{3.7}$$

Equations (3.5) and (3.6) are solved to obtain

$$\bar{F} = \frac{-2(k - \mathrm{i}\,\omega R_-)(1 - \mathrm{i}\,\delta Q_F)C_-}{\Delta(k,\omega)}, \tag{3.8}$$

where

$$Q_F = \left[\frac{1+q}{q}\ln(1+q) + \frac{1}{2}l\mathscr{D}\right]\omega + (q+1)\left[\frac{1}{q}\ln(1+q) + \frac{1}{2}l\mathscr{D}\right]\mathrm{i}k, \tag{3.9}$$

$$\Delta(k,\omega) = \mathscr{A}\omega^2 + \mathrm{i}\,\mathscr{B}\omega - \mathscr{C}, \tag{3.10}$$

with

$$\left.\begin{aligned} \mathscr{A} &= \left\{(R_+ + R_-) + \delta\left[\frac{(q+2)}{q}\ln(1+q) + l\mathscr{D}\right]k\right\}, \\ \mathscr{B} &= 2k\left\{1 + \delta\left[\frac{2(q+1)}{q}\ln(1+q) + \frac{1}{2}(q+2)l\mathscr{D}\right]k\right\}, \\ \mathscr{C} &= -(qk^2 + k(R_+ - R_-)G)\left[1 + \delta\frac{(1+q)}{q}\ln(1+q)k\right] \\ &\quad + \delta k^3\left[q + (q+1)l\mathscr{D} + \frac{(q+1)(q+2)}{q}\ln(1+q)\right]. \end{aligned}\right\} \tag{3.11}$$

Note that the result is independent of the Prandtl number, Pr. Obviously, a regular perturbation procedure can be employed to compute the $O(\epsilon^2)$ nonlinear response.

4. Acoustic field of a wrinkling flame

4.1. *Non-resonant case*

For a vortical disturbance in the form of a single spatial component (3.2), the flame wrinkles so that its total surface area increases. However, the transversely averaged flame surface area is independent of time, and as a result no longitudinal acoustic pressure is generated in this special case. Hereafter, we assume that the vortical disturbance consists of four spatial components:

$$\boldsymbol{k}^\dagger = (k_2, k_3), \quad \bar{\boldsymbol{k}}^\dagger = (k_2, -k_3), \quad -\bar{\boldsymbol{k}}^\dagger = (-k_2, k_3), \quad -\boldsymbol{k}^\dagger = (-k_2, -k_3), \tag{4.1}$$

the supposition of which represents standing waves in both transverse directions. The flame position F takes the form

$$F = \bar{F}\cos(k_2\eta)\cos(k_3\zeta)\,\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{c.c.}, \tag{4.2}$$

with $\bar{F}$ being still given by (3.8), since the flame responds to each mode independently. Now the transversely averaged flame area oscillates periodically with time t, with a frequency 2ω. On using (2.22), the unsteady heat release rate can be computed as

$$\mathscr{J} = \epsilon^2\frac{8qk^2(k-\mathrm{i}\,\omega R_-)^2}{\Delta^2(k,\omega)}(1-\mathrm{i}\,\delta Q_F)^2(1+\mathrm{i}\,\omega l\mathscr{D}\delta)^2C_-^2\,\mathrm{e}^{2\,\mathrm{i}\,\omega t} + \mathrm{c.c.} \equiv \epsilon^2\widehat{\mathscr{J}}\,\mathrm{e}^{2\,\mathrm{i}\,\omega t} + \mathrm{c.c.}, \tag{4.3}$$

through which the unsteady wrinkling flame radiates sound. Here the $O(\beta M)$ term in (2.22) is neglected, causing an $O(\beta M)$ error in the acoustic response, which should be quite acceptable.

In order to contrast with a ducted flame as well as to emphasize the fact that an acoustic field is an intrinsic part of a flame, we briefly consider the flame in open space. In this case, the emitted sound propagates outwards on either side of the flame, and so the solution for the acoustic pressure and velocity may be written as

$$(p_a,\ u_a) = \begin{cases} \epsilon^2(1,\ R_+^{-1/2})a_r\,\mathrm{e}^{2\,\mathrm{i}\,\omega(t-R_+^{1/2}\tilde{\xi})} + \mathrm{c.c.}, & \tilde{\xi} > 0, \\ \epsilon^2(1,\ -R_-^{-1/2})a_l\,\mathrm{e}^{2\,\mathrm{i}\,\omega(t+R_-^{1/2}\tilde{\xi})} + \mathrm{c.c.}, & \tilde{\xi} < 0. \end{cases}$$

On applying the conditions at $\tilde{\xi} = 0$, we find that

$$a_r = a_l = \widehat{\mathscr{J}}/(R_+^{-1/2} + R_-^{-1/2}).$$

Now return to the flame in the duct. A crucial difference from the open-flame case is that a duct supports acoustic modes with discrete eigenfrequencies so that a resonance may occur. On assuming that the frequency 2ω does not coincide with any of those eigenfrequencies, a steady-state response of $O(\epsilon^2)$ can be established. The solution may be written as

$$p_a = \epsilon^2\left\{a_r^\pm\,\mathrm{e}^{-2\,\mathrm{i}\,R_\pm^{1/2}\omega\tilde{\xi}} + a_l^\pm\,\mathrm{e}^{2\,\mathrm{i}\,R_\pm^{1/2}\omega\tilde{\xi}}\right\}\mathrm{e}^{2\,\mathrm{i}\,\omega t} + \mathrm{c.c.}, \tag{4.4}$$

$$u_a = \epsilon^2 R_\pm^{-1/2}\left\{a_r^\pm\,\mathrm{e}^{-2\,\mathrm{i}\,R_\pm^{1/2}\omega\tilde{\xi}} - a_l^\pm\,\mathrm{e}^{2\,\mathrm{i}\,R_\pm^{1/2}\omega\tilde{\xi}}\right\}\mathrm{e}^{2\,\mathrm{i}\,\omega t} + \mathrm{c.c.}, \tag{4.5}$$

where $\pm$ refers to the regions $\tilde{\xi} > 0$ and $\tilde{\xi} < 0$ respectively. The jump conditions across the hydrodynamic region, (2.21) and (2.22), yield

$$a_r^- + a_l^- = a_r^+ + a_l^+\ , \quad a_r^- - a_l^- - \left(\frac{R_-}{R_+}\right)^{1/2}(a_r^+ - a_l^+) = -R_-^{1/2}\widehat{\mathscr{J}}. \tag{4.6}$$

From (4.4), (4.5) and (2.23) it follows that

$$a_r^- \,\mathrm{e}^{2\,\mathrm{i}\,R_-^{1/2}\sigma\omega L} - a_l^- \,\mathrm{e}^{-2\,\mathrm{i}\,R_-^{1/2}\sigma\omega L} = 0, \quad a_r^+ \,\mathrm{e}^{-2\,\mathrm{i}\,R_+^{1/2}(1-\sigma)\omega L} + a_l^+ \,\mathrm{e}^{2\,\mathrm{i}\,R_+^{1/2}(1-\sigma)\omega L} = 0. \tag{4.7}$$

For the non-resonant case, for which $\Delta_s(2\omega;\sigma) \neq 0$, we find that

$$a_l^- = -\frac{\mathrm{i}\,\widehat{\mathscr{J}} R_+^{1/2} \tan\left(2R_+^{1/2}(1-\sigma)\omega L\right)}{2\Delta_s(2\omega;\sigma)\cos\left(2R_-^{1/2}\sigma\omega L\right)}\,\mathrm{e}^{2\,\mathrm{i}\,R_-^{1/2}\sigma\omega L}, \tag{4.8}$$

and the acoustic pressure level at the entrance (which we take to represent the sound field)

$$|p_a| = \frac{R_+^{1/2}|\widehat{\mathscr{J}}|\tan\left(2R_+^{1/2}(1-\sigma)\omega L\right)}{\Delta_s(2\omega;\sigma)\cos\left(2R_-^{1/2}\sigma\omega L\right)}, \tag{4.9}$$

where $\Delta_s(\omega)$ is given by (2.26). The ratio $|p_a|/\mathscr{J}$ corresponds to a transfer function.

Again, the analysis can easily be generalized to include the acoustic contribution from the $O(\epsilon^2)$ nonlinearly generated flame wrinkling.

4.2. *Resonant case*

The acoustic wave is merely a small passive by-product in the non-resonant case. However, if the condition

$$2\omega = \omega_k, \quad \text{i.e.} \quad \Delta_s(2\omega;\sigma) = 0, \tag{4.10}$$

is satisfied for some integer k and σ, the forcing is in resonance with the kth acoustic mode of the duct, generating strong acoustic pressure. The ensuing response depends on whether the flame is moving or if its mean position is fixed, as will be shown in the next two sections.

Although resonant excitation of acoustic modes occurs strictly for sinusoidal vortical disturbances with specific frequencies, it can be relevant for perturbations or ‘turbulence’, represented by (3.1) with a continuous spectrum. In this general case, the acoustic source $\mathscr{J}$ must be expressed at leading order as a Fourier integral

$$\mathscr{J} = \epsilon^2 \int_{-\infty}^{\infty} \widehat{\mathscr{J}}(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega. \tag{4.11}$$

Use of this in (2.31) shows that the induced acoustic velocities $u_a(0^\pm, t)$ have Fourier components at frequencies ω_k,

$$\widehat{u}_a(0^\pm, \omega_k) = \epsilon^2\left\{-\widehat{\mathscr{J}}(\omega_k)h(-0^\pm) - \left[\frac{\mathrm{i}\,\widehat{\mathscr{J}}(\omega_k)}{\Delta'(\omega_k;\sigma)}\right]t\right\}, \tag{4.12}$$

which amplify algebraically (linearly) in time, while components at all other frequencies remain bounded, suggesting that at large time the acoustic field may be dominated by duct modes.

As a demonstration, we consider an oncoming disturbance consisting of four spatial components as represented by (4.1) but with a continuous frequency spectrum, which is assumed to be Gaussian for simplicity. The corresponding C_- is given by

$$C_-(\tilde{\boldsymbol{k}}^\dagger, \omega) = [\delta(\tilde{\boldsymbol{k}}^\dagger - \boldsymbol{k}^\dagger) + \delta(\tilde{\boldsymbol{k}}^\dagger + \boldsymbol{k}^\dagger) + \delta(\tilde{\boldsymbol{k}}^\dagger - \bar{\boldsymbol{k}}^\dagger) + \delta(\tilde{\boldsymbol{k}}^\dagger + \bar{\boldsymbol{k}}^\dagger)]\Phi(\omega), \tag{4.13}$$

with

$$\Phi(\omega) = \frac{1}{\sigma_s} \exp\{-(\omega-\omega_p)^2/\sigma_s^2 - (\omega+\omega_p)^2/\sigma_s^2 + 4\omega_p^2/\sigma_s^2\}, \tag{4.14}$$

where $\pm\omega_p$ represents the peak frequencies and σ_s is a measure of the bandwidth; statistically, σ_s^{-1} represents the correlation time of the oncoming disturbance. The solution for F may be written as (cf. (4.15))

$$F = \cos(k_2\eta)\cos(k_3\zeta) \int_{-\infty}^{\infty} \widehat{F}(\omega)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}\omega, \tag{4.15}$$

where $\widehat{F} = \mathscr{T}(\omega)\Phi(\omega)$ with $\mathscr{T}(\omega)$ being given by the right-hand side of (3.8) with $C_- = 1$. It follows from (2.22) and (4.11) that

$$\widehat{\mathscr{J}}(\omega) = \frac{qk^2}{\pi} \int_{-\infty}^{\infty} \widehat{F}(\tilde{\omega})\widehat{F}(\omega-\tilde{\omega})\, \mathrm{d}\tilde{\omega}.$$

The expression for $\widehat{\mathscr{J}}$ involves in general a convolution of $\mathscr{T}$ with itself but can be approximated as

$$\widehat{\mathscr{J}}(\omega) = \frac{qk^2}{\sqrt{2\pi}\sigma_s} \mathscr{T}^2(\omega/2) \exp\left\{ -\frac{1}{2}(\omega - 2\omega_p)^2/\sigma_s^2 - \frac{1}{2}(\omega + 2\omega_p)^2/\sigma_s^2 + 8\omega_p^2/\sigma_s^2 \right\}, \tag{4.16}$$

in the limit $\sigma_s \ll O(1)$. Now assume that the resonant condition $2\omega_p = \omega_k$ ($k = 1$ say) is satisfied. Inserting (4.11) with (4.16) into (2.31), we find that for $1 \ll t \ll \sigma_s^{-1}$,

$$u_a(0^{\pm}, t) \sim -\epsilon^2 \left\{ \frac{\mathscr{T}^2(\omega_k/2)}{\Delta_s'(\omega_k;\sigma)} (\mathrm{i}\, \mathrm{e}^{\mathrm{i}\omega_k t} + \mathrm{c.c.}) \right\} t. \tag{4.17}$$

The above analysis can be generalized to an arbitrary spectrum that exhibits discrete peaks at acoustic frequencies with their bandwidth $\sigma_s \ll O(1)$. Acoustic modes can emerge from a perturbation possessing such a spectrum. Even more generally, the resonance-induced algebraic growth occurs as long as $(1/t)\int_0^t \mathscr{J}(\tau)\, \mathrm{e}^{\mathrm{i}\omega_k \tau}\, \mathrm{d}\tau$ attains an $O(1)$ asymptote for a sufficiently large time t, as may be deduced from (2.31). For an arbitrary vortical disturbance with a long correlation time scale $\sigma_s^{-1} \gg O(1)$, this requirement is likely to be met for $1 \ll t \ll \sigma_s^{-1}$. Given that the acoustic field excited by such a disturbance with a continuous spectrum approaches asymptotically that generated by sinusoidal perturbations, one may expect essential features of the ensuing flame-acoustic interaction to be captured by an idealized model for a vortical disturbance with a single resonant frequency.

Finally, we note that if the $O(\epsilon^2)$ nonlinear hydrodynamic fluctuation is included, it would contribute to the acoustic source an $O(\epsilon^3)$ correction, whose dependence on time is $\mathrm{e}^{3\mathrm{i}\omega_k t}$. Unlike the leading-order source (4.11), this small correction is not in resonance with any acoustic mode, suggesting that the $O(\epsilon^2)$ nonlinear hydrodynamic fluctuation would play a secondary role in the flame-acoustic interaction, and its omission should not alter the qualitative behaviours that we shall describe.

5. Resonant interaction between flame and spontaneously generated acoustic modes: flame with a fixed mean position

If the mean velocity of the fresh mixture equals the laminar flame speed, then the mean position of the flame remains fixed with respect to the laboratory coordinate, even though it oscillates sinusoidally with both time and space. The resonance condition (4.10) holds all the time. We will show in this section that the flame-acoustic resonance causes the entire system to evolve through two distinct regimes of development.

5.1. *Regime I: back action of the induced acoustic field on the flame*

Owing to the resonant response of the duct mode to the wrinkling flame, the intensity of the radiated acoustic field amplifies proportionally to time t, i.e. $u_a \sim \epsilon^2 t$. It acquires an $O(1)$ amplitude when $t \sim O(\epsilon^{-2})$, by which stage the acceleration of the acoustic field acts back on the flame through the R-T effect, and the flame and its induced acoustic field become mutually coupled. To describe the evolution towards and during this regime, we introduce the slow (long) time variable

$$\tau = \epsilon^2 t. \tag{5.1}$$

Due to the back action of the induced acoustic acceleration on the flame, the solution for the hydrodynamic field is no longer purely sinusoidal in time but must have a general dependence on time t. The dependence on the transverse variables nevertheless remains sinusoidal, since the flame and its hydrodynamic field remains linear. Thus the solution may be sought of the form

$$\begin{Bmatrix} U \\ \boldsymbol{V} \\ P \\ F \end{Bmatrix} = \epsilon \begin{Bmatrix} \tilde{U}(\xi, t) \\ \tilde{\boldsymbol{V}}(\xi, t) \\ \tilde{P}(\xi, t) \\ \alpha(t) \end{Bmatrix} \mathrm{e}^{\mathrm{i}(k_2\eta + k_3\zeta)} + \cdots, \tag{5.2}$$

where the other spatial Fourier components with $\bar{\boldsymbol{k}}^{\dagger}$, $(-\bar{\boldsymbol{k}}^{\dagger})$ and $(-\boldsymbol{k}^{\dagger})$ are not written out explicitly. It suffices to consider one of them, $\boldsymbol{k}^{\dagger}$, as representative, since all behave similarly. All four have to be taken into account when calculating $\mathscr{J}$, of course.

The $O(\delta)$ corrections to the hydrodynamic field and to the jump conditions are not central to the flame-acoustic resonance. For simplicity, we present the derivation using the inviscid version of (2.37) and the leading-order approximation for the jumps across the flame zone. The extension to include the $O(\delta)$ corrections in (2.37) and in jumps is relegated to the appendix. With $O(\delta)$ viscous terms in (2.37) being neglected, the solution can be found as

$$\tilde{U} = \begin{cases} \phi_-(t)\,\mathrm{e}^{k\xi} + C_- \exp\{\mathrm{i}\,\omega(R_-\xi - t)\}, & \xi < 0, \\ \phi_+(t)\,\mathrm{e}^{-k\xi} + C_+(t - R_+\xi), & \xi > 0, \end{cases} \tag{5.3}$$

$$\tilde{\boldsymbol{V}} = \begin{cases} \dfrac{\mathrm{i}\,\boldsymbol{k}^{\dagger}}{k}\phi_-(t)\,\mathrm{e}^{k\xi} + \boldsymbol{D}_- \exp\{\mathrm{i}\,\omega(R_-\xi - t)\}, & \xi < 0, \\ -\dfrac{\mathrm{i}\,\boldsymbol{k}^{\dagger}}{k}\phi_+(t)\,\mathrm{e}^{-k\xi} + \boldsymbol{D}_+(t - R_+\xi), & \xi > 0, \end{cases} \tag{5.4}$$

$$\tilde{P} = \begin{cases} -\dfrac{1}{k}(R_-\phi_-' + k\phi_-)\,\mathrm{e}^{k\tilde{\xi}}, & \xi < 0, \\ \dfrac{1}{k}(R_+\phi_+' - k\phi_+)\,\mathrm{e}^{-k\tilde{\xi}}, & \tilde{\xi} > 0, \end{cases} \tag{5.5}$$

where C_+ and $\boldsymbol{D}_+$ are arbitrary functions. It follows from the continuity equation that

$$-R_+C_+'(t) + \mathrm{i}\,\boldsymbol{k}^{\dagger} \cdot \boldsymbol{D}_+ = 0, \tag{5.6}$$

while the jump conditions give rise to the relations

$$\left.\begin{aligned}(R_+\phi_+' - k\phi_+) &= -(R_-\phi_-' + k\phi_-) + k\{(R_+ - R_-)G - \Delta p_{a,\xi}\}\alpha,\\ \phi_+ + C_+ &= \phi_- + C_-\exp\{-\mathrm{i}\,\omega t\},\\ -\frac{\mathrm{i}\,\boldsymbol{k}^\dagger}{k}\phi_+ + \boldsymbol{D}_+ &= \frac{\mathrm{i}\,\boldsymbol{k}^\dagger}{k}\phi_- + \boldsymbol{D}_-\exp\{-\mathrm{i}\,\omega t\} - \mathrm{i}\,\boldsymbol{k}^\dagger q\alpha,\\ \alpha' &= \phi_- + C_-\exp\{-\mathrm{i}\,\omega t\}.\end{aligned}\right\} \tag{5.7}$$

After eliminating $\phi_\pm$, C_+ and $\boldsymbol{D}_+$, we obtain

$$(R_+ + R_-)\alpha''(t) + 2k\,\alpha'(t) - \{qk^2 + k((R_+ - R_-)G - \Delta p_{a,\xi}(t))\}\alpha = \mathscr{N}_0\,\mathrm{e}^{-\mathrm{i}\,\omega t} + \text{c.c.}, \tag{5.8}$$

where $\Delta p_{a,\xi}$ is given by (2.45) and

$$\mathscr{N}_0 = 2(k - \mathrm{i}\,\omega R_-)C_-.$$

Note that the final result (5.8) is independent of the polarity of the wave disturbance, which means that the same result can be derived if any of the other three components is considered. Equation of the form (5.8) was derived by Markstein & Squire (1955) for an externally imposed acoustic field. In the absence of any oncoming vortical disturbance and acoustic feedback, it reduces to the one governing the D-L instability.

From (2.22), the jump in the acoustic velocity is obtained as $\mathscr{J} = \epsilon^2\widehat{\mathscr{J}}\,\mathrm{e}^{2\,\mathrm{i}\,\omega t} + \text{c.c.}$, with

$$\widehat{\mathscr{J}} = 2qk^2(\pi/\omega)\int_0^{\omega/\pi}\alpha^2(t)\,\mathrm{e}^{-2\,\mathrm{i}\,\omega t}\,\mathrm{d}\,t + \frac{1}{2}q(\beta M/\epsilon^2)\tilde{h}, \tag{5.9}$$

where the second term represents the acoustic instability mechanism due to the acoustic pressure modifying the burning rate. This effect is negligible when

$$\epsilon^2 \gg \beta M, \tag{5.10}$$

but it is retained here for completeness.

For $\tau = O(1)$, the solution for the acoustic field expands as

$$p_a = B(\tau)p_{a,1} + \epsilon^2 p_{a,2} + \text{ c.c.} + \cdots, \quad u_a = B(\tau)u_{a,1} + \epsilon^2 u_{a,2} + \text{ c.c.} + \cdots, \tag{5.11}$$

where $B(\tau)$ represents the amplitude of the acoustic mode. At leading order, $p_{a,1}$ and $u_{a,1}$ satisfy (2.20), and they have the solution

$$\left.\begin{aligned}p_{a,1} &= \mathrm{e}^{2\,\mathrm{i}\,\omega t}\left\{a_r^\pm\,\mathrm{e}^{-2\,\mathrm{i}\,R_\pm^{1/2}\omega\xi} + a_l^\pm\,\mathrm{e}^{2\,\mathrm{i}\,R_\pm^{1/2}\omega\xi}\right\} \equiv \hat{p}_{a,1}\,\mathrm{e}^{2\,\mathrm{i}\,\omega t},\\ u_{a,1} &= \mathrm{e}^{2\,\mathrm{i}\,\omega t}\,R_\pm^{-1/2}\left\{a_r^\pm\,\mathrm{e}^{-2\,\mathrm{i}\,R_\pm^{1/2}\omega\xi} - a_l^\pm\,\mathrm{e}^{2\,\mathrm{i}\,R_\pm^{1/2}\omega\xi}\right\} \equiv \hat{u}_{a,1}\,\mathrm{e}^{2\,\mathrm{i}\,\omega t},\end{aligned}\right\} \tag{5.12}$$

where the constants $a_r^\pm$ and $a_l^\pm$ satisfy the homogeneous version of (4.6) and (4.7), i.e. with the forcing term $\widehat{\mathscr{J}} = 0$. The eigenfunction is normalized such that

$$\left.\begin{aligned}a_l^- &= \mathrm{e}^{2\,\mathrm{i}\,R_-^{1/2}\sigma\omega L}, \quad a_l^+ = \frac{1}{2}\left(1 + \sqrt{\frac{R_+}{R_-}}\right)\mathrm{e}^{-2\,\mathrm{i}\,R_-^{1/2}\sigma\omega L} + \frac{1}{2}\left(1 - \sqrt{\frac{R_+}{R_-}}\right)\mathrm{e}^{2\,\mathrm{i}\,R_-^{1/2}\sigma\omega L},\\ a_r^- &= \mathrm{e}^{-2\,\mathrm{i}\,R_-^{1/2}\sigma\omega L}, \quad a_r^+ = \frac{1}{2}\left(1 - \sqrt{\frac{R_+}{R_-}}\right)\mathrm{e}^{-2\,\mathrm{i}\,R_-^{1/2}\sigma\omega L} + \frac{1}{2}\left(1 + \sqrt{\frac{R_+}{R_-}}\right)\mathrm{e}^{2\,\mathrm{i}\,R_-^{1/2}\sigma\omega L}.\end{aligned}\right\} \tag{5.13}$$

Inserting (5.11) into (2.20), at $O(\epsilon^2)$ we have

$$R\frac{\partial^2 p_{a,2}}{\partial t^2}-\frac{\partial^2 p_{a,2}}{\partial\tilde{\xi}^2}=-4\,\mathrm{i}\,\omega RB'(\tau)p_{a,1},\quad R\frac{\partial u_{a,2}}{\partial t}=-\frac{\partial p_{a,2}}{\partial\tilde{\xi}}-RB'(\tau)u_{a,1},\tag{5.14}$$

whose solutions are found to be

$$p_{a,2}=\mathrm{e}^{2\,\mathrm{i}\,\omega t}\Big[\big(b_r^{\pm}\,\mathrm{e}^{-2\,\mathrm{i}\,R_{\pm}^{1/2}\omega\tilde{\xi}}+b_l^{\pm}\,\mathrm{e}^{2\,\mathrm{i}\,R_{\pm}^{1/2}\omega\tilde{\xi}}\big)-R_{\pm}^{1/2}B'\tilde{\xi}\big(a_r^{\pm}\,\mathrm{e}^{-2\,\mathrm{i}\,R_{\pm}^{1/2}\omega\tilde{\xi}}-a_l^{\pm}\,\mathrm{e}^{2\,\mathrm{i}\,R_{\pm}^{1/2}\omega\tilde{\xi}}\big)\Big],\tag{5.15}$$

$$u_{a,2}=\mathrm{e}^{2\,\mathrm{i}\,\omega t}\Big[R_{\pm}^{-1/2}\big(b_r^{\pm}\,\mathrm{e}^{-2\,\mathrm{i}\,R_{\pm}^{1/2}\omega\tilde{\xi}}-b_l^{\pm}\,\mathrm{e}^{2\,\mathrm{i}\,R_{\pm}^{1/2}\omega\tilde{\xi}}\big)-B'\tilde{\xi}\big(a_r^{\pm}\,\mathrm{e}^{-2\,\mathrm{i}\,R_{\pm}^{1/2}\omega\tilde{\xi}}+a_l^{\pm}\,\mathrm{e}^{2\,\mathrm{i}\,R_{\pm}^{1/2}\omega\tilde{\xi}}\big)\Big].\tag{5.16}$$

Applying the boundary condition at the two ends and the jump condition across the flame,

$$[p_{a,2}]=0,\quad [u_{a,2}]=\widehat{\mathscr{J}}\,\mathrm{e}^{2\,\mathrm{i}\,\omega t}\quad \text{at}\quad \tilde{\xi}=0,\tag{5.17}$$

we obtain an algebraic system (cf. WMMP),

$$\boldsymbol{M}\,\boldsymbol{b}=\boldsymbol{d},\tag{5.18}$$

where $\boldsymbol{M}$ is given by (4.36) of WWMP; $\boldsymbol{b}=(b_r^-,b_l^-,b_r^+,b_l^+)^{\mathrm{T}}$; and

$$\boldsymbol{d}=(0,-R_-^{1/2}\widehat{\mathscr{J}},-2R_-^{1/2}a_l^-\,\mathrm{e}^{-2\,\mathrm{i}\,R_-^{1/2}\sigma\omega l}\,\sigma LB',-2R_+^{1/2}a_l^+\,\mathrm{e}^{2\,\mathrm{i}\,R_+^{1/2}(1-\sigma)\omega L}(1-\sigma)LB')^{\mathrm{T}}.\tag{5.19}$$

The matrix $\boldsymbol{M}$ is singular owing to (2.27). A solvability condition is therefore required, and this leads to the amplitude equation for the acoustic mode

$$B'(\tau)=\chi_a(\pi/\omega)\int_0^{\omega/\pi}\alpha^2(t;\tau)\,\mathrm{e}^{-2\,\mathrm{i}\,\omega t}\,\mathrm{d}t+m_pB,\tag{5.20}$$

where

$$\chi_a=qk^2\Lambda/(L\sin(2R_-^{1/2}\sigma\omega L)),\quad m_p=\frac{\beta M(\gamma-1)q\Lambda}{2\epsilon^2L}\cot(2R_-^{1/2}\sigma\omega L),\tag{5.21}$$

with Λ being the same as (4.41) of WWMP, provided that ω is replaced by 2ω. Now from (5.11)–(5.13), one finds that

$$u_{a,t}(0^-,t)=4\omega R_-^{1/2}\sin(2R_-^{1/2}\sigma\omega L)B(\tau)\,\mathrm{e}^{2\,\mathrm{i}\,\omega t}+\text{c.c.},\tag{5.22}$$

$$\Delta p_{a,\xi}(t)=-4\omega\left(\frac{R_+}{R_-}-1\right)R_-^{1/2}\sin\left(2R_-^{1/2}\sigma\omega L\right)B(\tau)\,\mathrm{e}^{2\,\mathrm{i}\,\omega t}+\text{c.c.}\equiv-\chi_cB(\tau)\,\mathrm{e}^{2\,\mathrm{i}\,\omega t}+\text{c.c.}\tag{5.23}$$

Inserting (5.23) into (5.8) yields a forced Mathieu equation

$$(R_++R_-)\alpha''(t)+2k\,\alpha'(t)-\{qk^2+k(R_+-R_-)G+k\chi_c(B(\tau)\,\mathrm{e}^{2\,\mathrm{i}\,\omega t}+\text{c.c.})\}\alpha=\mathscr{N}_0\,\mathrm{e}^{-\mathrm{i}\,\omega t}+\text{c.c.}\tag{5.24}$$

governing the flame front and in turn the hydrodynamic field. Here a prime denotes differentiation with respect to t, but the function $\alpha(t;\tau)$ also depends on the slow time variable τ due to the presence of the term proportional to $B(\tau)$, which represents the acoustic feedback effect. A generalization of (5.24) by including $O(\delta)$ corrections is given by (A 15) in the appendix.

The integro-differential system (5.20) and (5.24) (or its extension (A 15)) describes the mutual interaction between the flame and the spontaneously induced acoustic field. It couples the fast dynamics of flame wrinkling and the slow dynamics of the sound intensity, which evolve over a long time scale under the accumulated effect of the wrinkling flame. The appropriate initial condition for α and B can be prescribed as

$$\alpha \to \frac{\mathscr{N}_0}{\Delta_0}\,\mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.}, \qquad B \to \chi_a\big(\mathscr{N}_0^{*2}/\Delta_0^{*2}\big)\tau \quad \text{as} \quad \tau \to 0, \tag{5.25}$$

in order to match asymptotically with the previous stage, where the acoustic back action on the flame is negligible. Here

$$\Delta_0(k,\omega) = (R_+ + R_-)\omega^2 + 2\,\mathrm{i}\,k\,\omega + [qk^2 + k(R_+ - R_-)G]. \tag{5.26}$$

Note that the acoustic intensity B is zero at $\tau = 0$, implying that the system starts from an essentially silent initial state. Any acoustic fluctuation that arises during the course of the evolution is entirely generated by the flame. According to (2.35) and the second equation in (2.34), the acoustic velocity fluctuation would cause the flame position to vibrate with a frequency twice that of flame wrinkling, as observed in experiments (Markstein 1953). This is however a dynamically passive effect for a freely propagating flame under consideration.

5.2. *Regime II: initiation of parametric instability of flame and its coupling with the self-induced sound*

Since (5.24) involves no derivative with respect to the slow variable τ, the acoustic intensity B plays the role of a 'parameter', as far as the flame response is concerned. This implies that the flame responds to the instantaneous acoustic pressure in a quasi-steady fashion. It is known that for a given ω there is a threshold pressure amplitude $B_c = B_c(\omega)$ such that a subharmonic parametric resonance can occur in the homogeneous system (Markstein 1955; Searby & Rochwerger 1991); that is the homogeneous system admits a non-trivial solution even in the absence of any forcing. It is therefore expected that as $B(\tau) \to B_c$, the forcing starts to resonate with the parametric instability mode so that the flame response $\alpha(\tau)$ would develop a singularity at a finite time τ_c. The system (5.20) and (5.24) cannot be integrated beyond τ_c.

In the vicinity of τ_c, a new regime emerges because a well-defined subharmonic parametric instability mode is excited. The whole system evolves over a time scale much shorter than $O(\epsilon^{-2})$, and as a result the response of the flame to the acoustic field is no longer quasi-steady. Rather it is affected by the rate of change of the parametric instability mode so that the new phase of the evolution may be referred to as the 'non-equilibrium' regime. A scaling argument, based on balancing the rate of change of the parametric instability mode and the back action of the acoustic field, suggests that the new regime commences when $(\tau - \tau_c) \sim O(\epsilon^{3/2})$, and so we introduce

$$\tau = \tau_c + \epsilon^{3/2}\hat{\tau}. \tag{5.27}$$

The solution for the acoustic field expands as

$$p_a = B(\tau)p_{a,1} + \epsilon p_{a,2} + \cdots, \quad u_a = B(\tau)u_{a,1} + \epsilon u_{a,2} + \cdots, \tag{5.28}$$

where the intensity of the sound deviates from the threshold value B_c by $O(\epsilon^{1/2})$ amount so that B is expressed as

$$B = B_c + \epsilon^{1/2}\widehat{B}(\hat{\tau}), \quad \text{with} \quad \widehat{B} = O(1). \tag{5.29}$$

The amplitude of the flame, F, rises to $O(\epsilon^{1/2})$, and the solution for $\alpha(t)$ can be written as

$$\alpha(t;\hat{\tau}) = \epsilon^{-1/2}(A(\hat{\tau})\alpha_0(t) + \text{ c.c.}) + \alpha_1 + \cdots. \tag{5.30}$$

At leading order, α_0 is a marginal parametric instability mode, with a global amplitude $A(\hat{\tau})$. It is governed by the equation,

$$\mathscr{L}\alpha_0 = 0, \tag{5.31}$$

where

$$\mathscr{L} = \frac{d^2}{dt^2} + \left\{\frac{2k}{R_+ + R_-}\right\}\frac{d}{dt} - \left\{\frac{qk^2 + k(R_+ - R_-)G + k\chi_c(B_c\,e^{2i\omega t} + \text{c.c.})}{R_+ + R_-}\right\}. \tag{5.32}$$

For the subharmonic resonance, the solution for α_0 can be written as

$$\alpha_0 = e^{-i\omega t}\sum_{n=-\infty}^{\infty}\widehat{\alpha}_n\,e^{-2in\omega t}.$$

The amplitude is determined by considering α_1, which satisfies

$$\mathscr{L}\alpha_1 = -A'\left(2\alpha_0' + \frac{2k}{R_+ + R_-}\alpha_0\right) + \frac{k\chi_c}{R_+ + R_-}(\widehat{B}\,e^{2i\omega t} + \text{c.c.})\alpha_0 + \frac{\mathscr{N}_0}{R_+ + R_-}\,e^{-i\omega t}. \tag{5.33}$$

Since the homogeneous system admits an eigensolution, an appropriate solution for α_1 exists only if the right-hand side of (5.33) satisfies a solvability condition, which as usual can be imposed by introducing the adjoint operator of (5.32):

$$\frac{d^2\alpha^\dagger}{dt^2} - \left\{\frac{2k}{R_+ + R_-}\right\}\frac{d\alpha^\dagger}{dt} - \left\{\frac{qk^2 + k(R_+ - R_-)G + k\chi_c(B_c\,e^{2i\omega t} + \text{c.c.})}{R_+ + R_-}\right\}\alpha^\dagger = 0. \tag{5.34}$$

Now multiplying the adjoint function $\alpha^\dagger$ with both sides of (5.33) and integrating by parts, we obtain

$$A'(\hat{\tau}) = \gamma_a(\widehat{B} + \widehat{B}^*)A + \hat{\mathscr{N}}_0, \tag{5.35}$$

where

$$\gamma_a = -\frac{k\chi_c}{R_+ + R_-}\int_0^{2\pi/\omega} e^{2i\omega t}\,\alpha_0\alpha^\dagger\,dt \bigg/ \int_0^{2\pi/\omega}\left\{2\alpha_0' + \frac{2k}{R_+ + R_-}\alpha_0\right\}\alpha^\dagger\,dt,$$

$$\hat{\mathscr{N}}_0 = \frac{\mathscr{N}_0}{R_+ + R_-}\int_0^{2\pi/\omega} e^{-i\omega t}\,\alpha^\dagger\,dt \bigg/ \int_0^{2\pi/\omega}\left\{2\alpha_0' + \frac{2k}{R_+ + R_-}\alpha_0\right\}\alpha^\dagger\,dt.$$

The flame drives the acoustic field via the jump $\mathscr{J}$. Inserting (5.30) into (2.22) yields

$$\mathscr{J} = 2\epsilon qk^2(A + A^*)^2\sum_{p=-\infty}^{\infty}\widehat{\alpha}_{-p}\widehat{\alpha}_p\,e^{2i\omega t} + \text{c.c.} + \cdots, \tag{5.36}$$

where only the Fourier component in resonance with the duct mode is written out explicitly. Substituting expansion (5.28) along with (5.29) into the acoustic equations, we obtain, from the solvability condition at $O(\epsilon)$, the amplitude equation for $\widehat{B}$,

$$\widehat{B}'(\hat{\tau}) = \gamma_b(A + A^*)^2, \tag{5.37}$$

where

$$\gamma_b = \chi_a\sum_{p=-\infty}^{\infty}\widehat{\alpha}_{-p}\widehat{\alpha}_p.$$

Equations (5.35) and (5.37) describe the interaction of a parametrically marginally unstable flame with its acoustic field. On anticipating that A forms an algebraic singularity towards the end of the previous regime, it can be shown that the structure consistent with (5.35) and (5.37) must be $A \sim \hat{\tau}^{-1/3}$ and $\widehat{B} \sim \hat{\tau}^{1/3}$. The asymptotic matching requirement means that the appropriate initial condition can be prescribed as

$$A \to a_0 \hat{\tau}^{-1/3}, \quad \widehat{B} \to b_0 \hat{\tau}^{1/3} \quad \text{as} \quad \hat{\tau} \to -\infty, \tag{5.38}$$

where the complex constants a_0 and b_0 are determined by

$$\gamma_a(b_0 + b_0^*)a_0 + \hat{\mathscr{N}}_0 = 0, \qquad \frac{1}{3}b_0 = \gamma_b(a_0 + a_0^*)^2. \tag{5.39}$$

It is worth noting that (5.35) and (5.37) are somewhat different from the generic amplitude equations, $A' = A^* B$ and $B' = A^2$, for subharmonic resonant waves (e.g. Craik 1985). This is due to the fact that the acoustic mode and the parametric instability mode of the flame are comprised of both left and right travelling waves.

6. Resonant interaction between flame and spontaneously generated acoustic modes: a moving flame

If the mean position of the flame is propagating, the 'parameter' denoting the flame position, σ, is a (slow) function of time. For definitiveness, assume that the flame is at the inlet when $t = 0$ say. Then σ and t are related by $\sigma = 1 - M(1 - U_-)t/L$. For a given ω, the resonance condition $\Delta_s(2\omega, \sigma_c) = 0$ could be satisfied only at a particular time t_c. Resonance is expected to take place within a time window centred at t_c. As the flame propagates, the resonance condition deteriorates. The resonance must therefore be of transient nature. Since the duct is assumed to be very long, $O(M^{-1}h^*)$, the characteristic frequency of the acoustic modes may not be altered significantly for a considerable period of time. A scaling argument shows that for $(t - t_c) \gg O(M^{-1/2})$, the duct responds in a quasi-steady manner, with the moving mean flame position playing the role of a parameter. The response exhibits the character of resonance for $(t - t_c) \sim O(M^{-1/2})$, but the relatively small and slow change of the flame position with time must be accounted for. The appropriate time variable describing the transient resonant phase is introduced by writing

$$t = t_c + M^{-1/2}\tau. \tag{6.1}$$

Through the transient resonance, the intensity of the sound is amplified by a factor $O(M^{-1/2})$, reaching $O(\epsilon^2 M^{-1/2})$. In order to allow for a possible impact of the induced sound on the flame, the Mach number M is scaled such that

$$M = m\epsilon^4 \quad \text{with} \quad m = O(1).$$

Without losing generality, we take $m = 1$.

The solution for the acoustic field may still be written as (5.11). The two ends of the duct correspond to $\tilde{\xi} = -\sigma_c L + M^{1/2}(1 - U_-)\tau$ and $\tilde{\xi} = (1 - \sigma_c)L + M^{1/2}(1 - U_-)\tau$ respectively. Thus the boundary conditions for $p_{a,2}$ are modified to

$$\left.\begin{aligned} u_{a,2} &= -\frac{\partial u_{a,1}}{\partial \tilde{\xi}}(1 - U_-)\tau \quad \text{at} \quad \tilde{\xi} = -\sigma_c L, \\ p_{a,2} &= -\frac{\partial p_{a,1}}{\partial \tilde{\xi}}(1 - U_-)\tau \quad \text{at} \quad \tilde{\xi} = (1 - \sigma_c)L. \end{aligned}\right\} \tag{6.2}$$

Imposition of the boundary and jump conditions leads to (5.18), with $\boldsymbol{d}$ now being replaced by

$$\boldsymbol{d} = \begin{bmatrix} 0 \\ -R_-^{1/2}\widehat{\mathscr{J}} \\ -2R_-^{1/2}a_l^- \,\mathrm{e}^{-2\mathrm{i}R_-^{1/2}\sigma_c\omega l}\{\sigma_c LB' - \mathrm{i}\,\omega(1-U_-)\tau B\} \\ -2R_+^{1/2}a_l^+ \,\mathrm{e}^{2\mathrm{i}R_+^{1/2}(1-\sigma_c)\omega L}\{(1-\sigma_c)LB' + \mathrm{i}\,\omega(1-U_-)\tau B\} \end{bmatrix}. \tag{6.3}$$

The solvability condition yields the evolution equation for B,

$$B'(\tau) = \mathrm{i}\,\chi_s\tau B + \chi_a(\pi/\omega)\int_0^{\omega/\pi}\alpha^2(t;\tau)\,\mathrm{e}^{-2\mathrm{i}\omega t}\,\mathrm{d}t + m_p B, \tag{6.4}$$

where

$$\chi_s = -\frac{2\omega}{L}\left(\frac{R_+}{R_-}-1\right)(1-U_-)\tan(2R_-^{1/2}\sigma_c\omega L)\Lambda. \tag{6.5}$$

The extra term $\mathrm{i}\,\chi_s\tau B$ in (6.4) represents the detuning effect caused by the movement of the mean flame position (cf. (5.20)).

The flame front continues being governed by (5.24) and is rewritten here:

$$(R_+ + R_-)\alpha''(t) + 2k\,\alpha'(t) - \{qk^2 + k(R_+ - R_-)G + k\chi_c(B(\tau)\,\mathrm{e}^{2\mathrm{i}\omega t} + \mathrm{c.c.})\}\alpha = \mathscr{N}_0\,\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{c.c.} \tag{6.6}$$

However, the appropriate initial condition becomes

$$\alpha \to \frac{\mathscr{N}_0}{\Delta_0}\,\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{c.c.}, \quad B \to \frac{\mathrm{i}\,\mathscr{N}_0^{*2}}{\chi_s\Delta_0^{*2}}\,\tau^{-1} \quad \text{as} \quad \tau \to -\infty, \tag{6.7}$$

in order to match with the solution in the pre-resonance stage.

Depending on the intensity of the oncoming vortical disturbance, the induced acoustic pressure may still reach the threshold B_c for the subharmonic resonance at a time τ_c. The evolution in its vicinity would still be described by (5.35)–(5.37), because over this short time scale, the position of the flame appears fixed to leading-order approximation.

We conclude the analysis part of this paper by reiterating that the simplified equations derived in §§ 5 and 6 apply only to intrinsically stable flames. In most practical and laboratory conditions, flames are often subjected to hydrodynamical and/or thermal-diffusive instabilities. Experiments (e.g. Searby 1992) indicate that the initial hydrodynamic instability may lead to a violent parametric instability. Developing a simplified mathematical theory capable of predicting this phenomenon is left for further research.

7. Numerical solutions

The results/equations derived in previous sections have been evaluated/solved numerically. Unless stated otherwise, the parameters used are as follows:

$$h^* = 10\ \mathrm{cm}, \quad l^* = 120\ \mathrm{cm}, \quad q = 5, \quad U_L = 10\ \mathrm{m\,s^{-1}},$$
$$D_{th}^* = 0.22\ \mathrm{cm^2\,s^{-1}}, \quad a^* = 340\ \mathrm{m\,s^{-1}}, \quad Le = 1.11, \quad \beta = 12, \quad \gamma = 1.4.$$

These parameters are close to the experimental condition of Searby (1992) but with a slower flame speed. They give rise to $\delta = 2.2\times10^{-3}$, Mach number $M = 2.94\times10^{-4}$ and Markstein number $Ma = 4$ (cf. Searby & Clavin 1986).

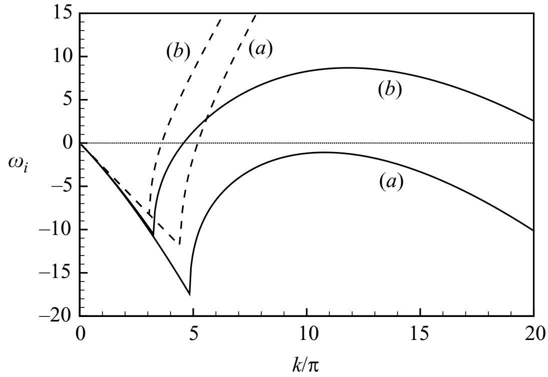

FIGURE 1. The growth rate of D-L instability ω_i versus the total transverse wavenumber k for (a) U_L = 10 cm s^{-1} and (b) U_L = 12 cm s^{-1}. Solid line: second-order theory; dashed-line: first-order theory.

Figure 1 shows growth rates of the D-L instability, predicted by the leading- and second-order theories, i.e. by

$$\Delta_0(k, \omega) = 0 \quad \text{and} \quad \Delta(k, \omega) = 0$$

respectively (see (3.10) and (5.26)). The leading-order approximation is valid for relatively small k, but yields an incorrect result of instability at large values of k for all $U_L \neq 0$. The second-order diffusive effect from the preheat zone plays a stabilizing role, especially at large k. This effect provides an intrinsic length scale, as reflected by the fact that there now exists the most unstable (or least stable) wavenumber. The instability is possible only for a finite band of wavenumers (see e.g. the case for U_L = 12 cm s^{-1}), and there exists a cutoff wavenumber beyond which all modes are stable. In the small-k limit, the stability is significantly influenced by the gravity, and ω has a non-zero real part ω_r (i.e. the modes are oscillatory but damped). The growth rate decreases with k, since the stabilizing effect of the gravity is proportional to k. It then increases and becomes positive eventually when the destabilizing effect of gas expansion takes over. The cusp (or 'corner') signals a switch from a complex ω to a purely imaginary ω. When the flame speed U_L = 10 cm s^{-1}, all D-L modes are damped; i.e. the flame is hydrodynamically stable. Thermal-diffusive instability is also absent because of the small deviation of the Lewis number from unity.

The response of the flame to perturbations of two representative transverse wavenumbers is shown in figure 2. Since any non-zero scaled longitudinal velocity of the oncoming perturbation $C_- \neq 0$ can be absorbed into the definition of ϵ, we take $C_- = 1$ in all the calculations for a fixed flame. For $k = 3\pi$, the response function exhibits a peak at a particular ω, reminiscent of that of a damped oscillator to harmonic forcing. This is expected, since the flame supports oscillatory modes with the relatively small k. For $k = 8\pi$, gravity plays a negligible role (see figure 1), and the flame response decreases monotonically with ω. The wrinkled flame drives sound waves in the duct due to the unsteady heat release associated with the surface-area change. The variation of the intensity of the induced acoustic pressure with the perturbation frequency is displayed in figure 3. As is expected, one observes a highly tuned response, with an infinite peak occurring at a discrete frequency ω such that 2ω coincides with a natural frequency of an acoustic duct mode; i.e. the resonant condition (4.10) is satisfied. Here only the resonance of the flame with the fundamental

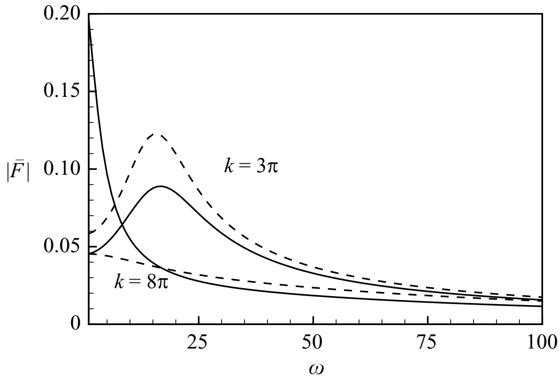

FIGURE 2. The flame response to vortical perturbations as measured by $|\bar{F}|$ versus the frequency ω of the perturbation with different transverse wavenumbers $k = 3\pi$ and $k = 8\pi$. Solid line: second-order result; dashed line: first-order result.

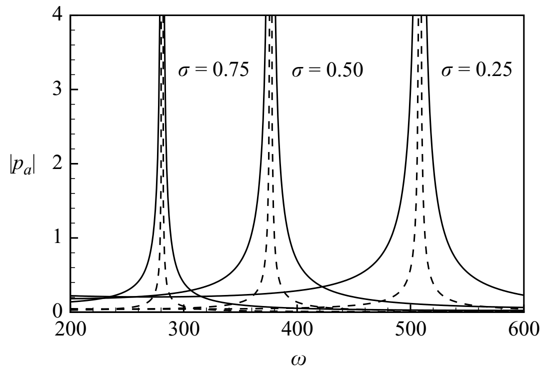

FIGURE 3. The acoustic pressure $|p_a|$ generated by a wrinkled flame versus the frequency ω of the perturbation. Three flame positions considered correspond to $\sigma = 0.25$, 0.5 and 0.75. Solid lines: $k = 8\pi$; dashed lines: $k = 3\pi$.

duct mode is shown, since resonance with higher modes is likely to be less effective due to their high frequencies.

The resonant case is of particular interest, since the accumulated intensification of the acoustic pressure arising from the resonance between the duct mode and the flame leads to a fully coupled stage, the first regime of which is governed by the evolution system (5.20) and (5.24) or (A 15). Equation (5.20) is integrated forward numerically by using a fifth-order Adam–Bashforth multiple-step scheme. The solution for α may be written as

$$\alpha(t;\tau) = \mathrm{e}^{-\mathrm{i}\omega t} \sum_{n=-\infty}^{\infty} \alpha_n(\tau)\, \mathrm{e}^{-2\mathrm{i}n\omega t},$$

where the reality of α requires $\alpha_{-(n+1)} = \alpha_n^*$. Substituting it into (5.24) shows that α_n for $n \geqslant 0$ satisfies

$$\Delta_0(n\omega)\alpha_n + k\chi_c(B^*(\tau)\alpha_{n-1} + B(\tau)\alpha_{n+1}) = -\mathscr{N}_0\delta_{n0}. \tag{7.1}$$

A truncated system would consist of simultaneous linear equations with a tridiagonal matrix, which can be solved at each time step by using LU factorization for a given

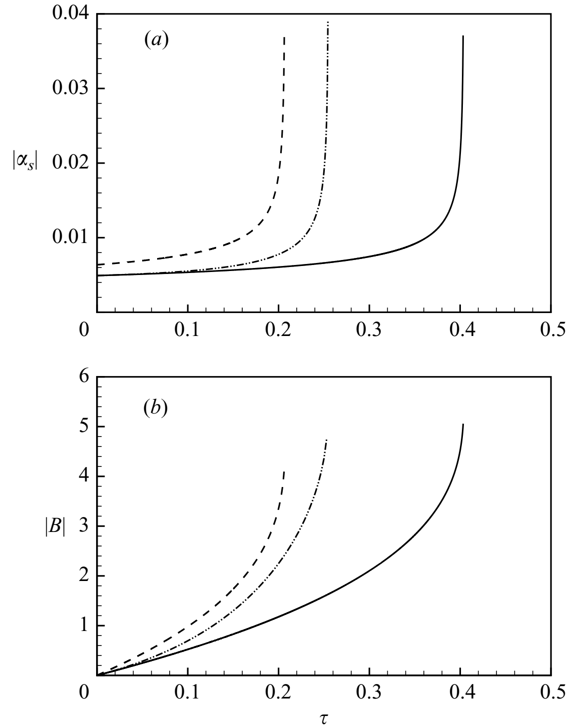

FIGURE 4. Evolution of flame wrinkling and the sound intensity in regime I of the two-way interaction for $k = 8\pi$: (*a*) $|\alpha_s(\tau)|$ versus τ and (*b*) $|B(\tau)|$ versus τ. Solid lines: second-order theory; dashed lines: first-order theory; dashed–dotted lines: secondary-order theory including the direct pressure effect ($m_p = 5.0$).

$B(\tau)$. The extended equation (A 15) is solved in a similar fashion, since it is of the same form as (5.24).

Figure 4 shows the evolution of flame wrinkling, measured by

$$\alpha_s(\tau) \equiv \left[\sum_{n=-\infty}^{\infty} |\alpha_n(\tau)|^2\right]^{1/2},$$

and the corresponding acoustic intensity B. Both the leading- and second-order results indicate that the back action of the sound on the flame via the R-T effect causes further wrinkling, which in turn produces more intense sound; the mutual interaction thus forms a positive feedback loop. The acoustic pressure intensity soon approaches the threshold amplitude for the subharmonic parametric instability within a finite time τ_c, at which point α_s develops a singularity. While B remains regular, its gradient B' becomes singular. This massively destabilizing effect of the acoustic feedback on the overall system appears generic and is observed at other parameter values, for example $k = 3\pi$ (in which case the singularity occurs at a later time, at about $\tau_c \approx 4$). Note that it takes place however weak the vortical perturbation is. When the pressure effect on the burning rate is included, the sound intensity and flame wrinkling amplify faster than otherwise. Despite this appreciable difference, the qualitative behaviour remains the same. Crucially, the formation of the singularity is due to the modulation of the flame surface area. It is also worth noting that the amplifying effect represented

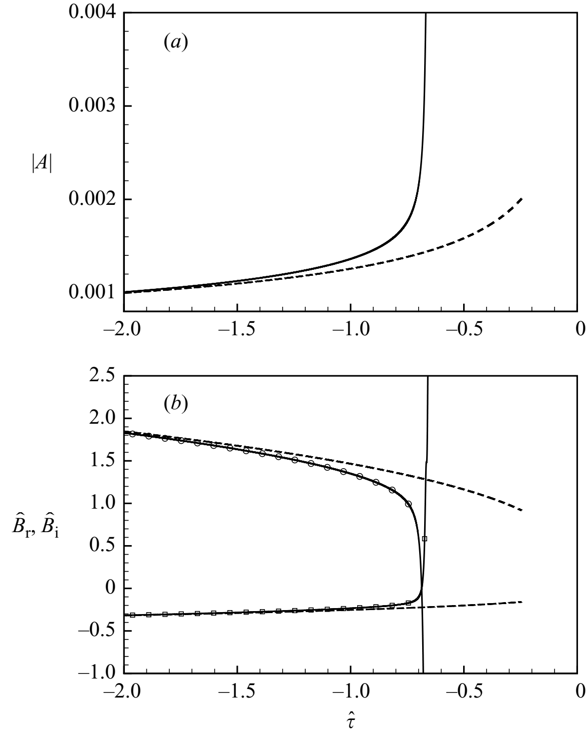

FIGURE 5. Evolution of the amplitude of the parametric instability mode and the sound intensity in regime II of the two-way interaction for $k = 8\pi$: (*a*) $|A(\hat{\tau})|$ versus $\hat{\tau}$ and (*b*) real and imaginary parts of $\widehat{B}(\hat{\tau})$, $\widehat{B}_r$ and $\widehat{B}_i$ versus $\hat{\tau}$. Solid lines: non-equilibrium response; dashed lines: asymptote (5.38) in regime I.

by $m_p B$ may partially be cancelled by the acoustic damping that is ignored. It was found in WWMP that neglecting $m_p B$ (along with the damping) gave a result in better agreement with experiments.

In the vicinity of τ_c, the subharmonic parametric instability mode of the flame is coupled with the acoustic mode. Their evolution is described by (5.35) and (5.37). The 'non-equilibrium' nature is reflected by the fact that the acoustic field controls the rate of change of the flame, rather than the flame amplitude itself (cf. (5.35) and (5.24)). Equations (5.35) and (5.37) are solved by using a 4th-order Rounge-Kutta method. Figure 5 shows the evolution of A, the amplitude of the parametric instability mode, and $\widehat{B}$, the acoustic intensity, over the shorter time variable $\hat{\tau}$. For $k = 8\pi$, the non-equilibrium effect further destabilizes the system, causing even more rapid amplification of flame wrinkling and sound wave. Both A and $\widehat{B}$ develop a new singularity at $\hat{\tau}_c < 0$, before the singularity in regime I is reached. It is easy to show that the singularity of the solution to (5.35) and (5.37) must be of the form

$$A \sim \hat{a}_0(\hat{\tau}_c - \hat{\tau})^{-1}, \quad B \sim \hat{b}_0(\hat{\tau}_c - \hat{\tau})^{-1} \quad \text{as} \quad \hat{\tau} \to \hat{\tau}_c, \tag{7.2}$$

where $\hat{a}_0$ and $\hat{b}_0$ satisfy

$$1 = \gamma_a(\hat{b}_0 + \hat{b}_0^*), \quad -\hat{b}_0 = \gamma_b(\hat{a}_0 + \hat{a}_0^*)^2. \tag{7.3}$$

It is worth pointing out that not only does this singularity occur at a shorter time scale, but also it is of a much severer type than the one in the previous regime.

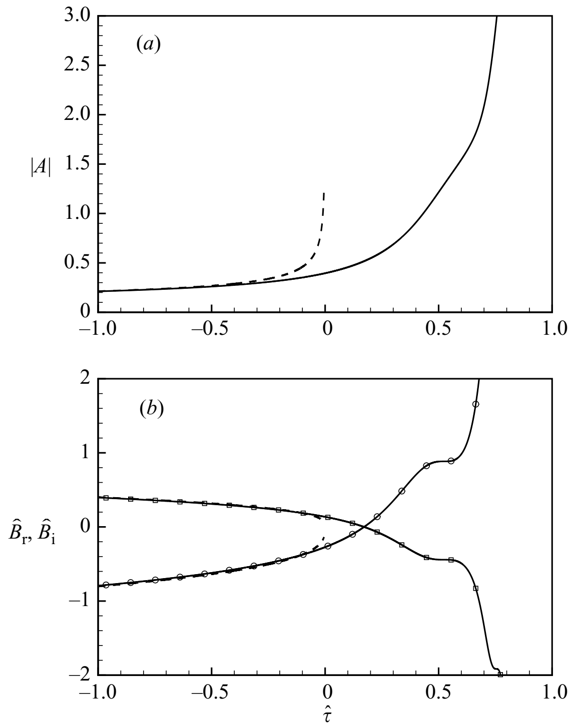

FIGURE 6. Evolution of the amplitude of the parametric instability mode and the sound intensity in regime II of the two-way interaction for $k = 3\pi$: (*a*) $|A(\hat{\tau})|$ versus $\hat{\tau}$ and (*b*) real and imaginary parts of $\widehat{B}(\hat{\tau})$, $\widehat{B}_r$ and $\widehat{B}_i$ versus $\hat{\tau}$. Solid lines: non-equilibrium response; dashed lines: asymptote (5.38) in regime I.

Interestingly, for $k = 3\pi$ shown in figure 6 the non-equilibrium effect initially plays a *stabilizing* role, in the sense that it first renders the amplification to be slower than in regime I, and moreover it is able to remove the singularity (5.38). However, this stabilizing effect is short lived. Eventually, the amplification becomes faster, and the solution, while bypassing the singularity (5.38), develops the same singularity as (7.2) at $\hat{\tau}_c > 0$. Note that the final singularity, (7.2) and (7.3), is unaffected by the external forcing term $\mathscr{N}$. This implies that on approaching the singularity the internal dynamics dominate, while the external perturbation, having led the system to the present stage by initiating first the acoustic-flame resonance and then the subharmonic parametric instability, has become largely irrelevant.

It is noted that in regime II, the self-nonlinear effect of the (parametric instability) mode, which typically contributes a Landau type of cubic nonlinear term $A|A|^2$ to the amplitude equation (Stuart 1960), does not affect the evolution because of the relatively small characteristic magnitude. There comes the question: would the increased amplitude through the singularity (7.2) lead to yet another regime in which the self-singularity is important? The question can be settled by a scaling argument as follows: As $\hat{\tau} \to \hat{\tau}_c$, the cubic nonlinear term is $O(\epsilon^{3/2}(\hat{\tau}_c - \hat{\tau})^{-3})$, while the rate of change of the flame amplitude $\epsilon^{1/2}\,\mathrm{d}\alpha/\mathrm{d}t \sim O(\epsilon(\hat{\tau}_c - \hat{\tau})^{-2})$. The two become comparable when

$$(\hat{\tau}_c - \hat{\tau}) \sim O(\epsilon^{1/2}).$$

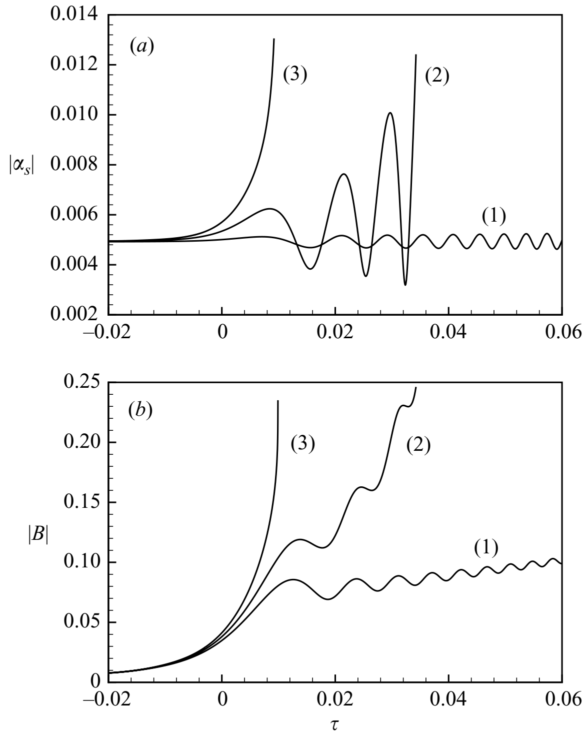

FIGURE 7. Evolution of flame wrinkling and the sound intensity during the transient resonance phase: (*a*) $|\alpha_s(\tau)|$ versus τ and (*b*) $|B(\tau)|$ versus τ. Curves (1), (2) and (3) represent the results for the oncoming velocities $C_- = 2$, 4 and 5 respectively.

Within the vicinity of $\hat{\tau}_c$ specified above, flame wrinkling α and the deviation of the acoustic intensity from B_c become $O(1)$, while the time scale of the evolution becomes comparable with the period of the acoustic and flame oscillation so that the notion of a wave envelope is no longer attainable. Both consequences indicate that the singularity (7.2) leads the system directly into a strongly nonlinear regime without going through any further intermediate weakly nonlinear phase. It may be speculated that in this final regime a state of self-sustained acoustic-flame oscillation is established.

Finally, we turn to the case of a moving flame and solve the evolution system (6.4)–(6.6), which describes the flame-acoustic interaction during the transient resonance phase. The result for $U_- = 0.9$ is shown in figure 7. Unlike the case of a fixed flame, the dynamics now depend on the amplitude of the oncoming perturbation, characterized by C_-. For a relatively small $C_- = 2$ (curves (1)), the amplification of flame wrinkling by the transient resonance is rather minimal. There is an appreciable gain in the induced sound, due to the accumulated action of flame wrinkling, and for the same reason, the sound wave amplitude B continues to amplify as τ increases, but the slow amplification suggests that B is unlikely to reach the threshold amplitude for subharmonic parametric resonance. An interesting feature is that both $|\alpha_s|$ and B exhibit a rather regular oscillatory behaviour in the post resonance phase. This is a beating phenomenon caused by the difference between the constant frequency of the vortical perturbation and the frequency of the duct mode which varies slowly with time for a moving flame. For the relatively strong perturbation $C_- = 5$ (curve 3) say, the induced pressure reaches B_c during the transient resonance phase so that subharmonic resonance is initiated, and the solution soon develops singularity (5.38) within a finite time. For a moderate $C_- = 4$, flame wrinkling and sound intensity undergo strong oscillations before terminating at the singularity. In any case, it may be expected that the transient resonance would lead to one or more 'episodes', during which the acoustic pressure rises to an appreciate level, while the flame position vibrates with a frequency twice that of flame wrinkling. These are in qualitative agreement with the observations summarized by Markstein (1953). The multiple episodes detected in experiments (figure 15 of Markstein 1953) are probably related to the modulated oscillatory response predicted here (figure 7*b*).

8. Discussions and conclusions

In this paper, we investigated the influence of vortical disturbances, which represent weak turbulence superimposed on the oncoming fresh mixture, on a stable planar premixed flame in a duct, with particular attention to their role in initiating and interfering the flame-acoustic coupling. Under the assumption that the vortical disturbance is of small amplitude and consists of a single Fourier component with a frequency (ω), systematic linear and weakly nonlinear approximations were made to derive from the general acoustic-flame interaction theory of WWMP a series of simplified equations, which describe the evolution of the flame-acoustic system and the dominant physics involved.

When a flame is wrinkled by the perturbation, the time-periodic heat release due to the surface-area change drives acoustic oscillations in the duct. The main focus of the present study was on the resonant case, in which 2ω coincides with the characteristic frequency of the fundamental acoustic mode of the duct so that the latter amplifies linearly with time t. The spontaneously generated acoustic field then acquires an $O(1)$ magnitude to act back on the flame via the unsteady R-T effect. Two distinct regimes of this two-way coupling were identified. In the first regime, the flame responds to the acoustic pressure in a quasi-steady fashion. The evolution was governed by a novel integro-differential system. Numerical solutions indicate that the mutual interaction massively enhances flame wrinkling and acoustic oscillation. In the case of a fixed flame, the acoustic intensity approaches the threshold amplitude for the subharmonic parametric instability, at which point flame wrinkling develops a singularity. In the second regime, the parametric instability was initiated and was coupled with the acoustic mode. The new coupled equations develop yet another singularity, which eventually takes the system to a final fully nonlinear stage. The above results provide a detailed first-principle description of the two-way dynamical interaction between a wrinkling flame and the spontaneously generated sound (Markstein 1953; Markstein & Squire 1955), which is believed to be of fundamental importance to combustion instability.

An upshot of the present analysis is that a suitable small-amplitude vortical perturbation may completely destabilize a flame that is hydrodynamically and thermal-diffusively stable. This remarkable destabilization mechanism involves three simultaneously operating elementary processes: flame wrinkling in a flow field, excitation of acoustic mode by unsteady heat release and the parametric instability induced by the acoustic pressure. Each of these has been studied separately and reasonably well understood. While those studies are invaluable, it is the *dynamic* interplay among all the three processes that is most instrumental to sustaining combustion instability. Illustrating this point is the parametric instability. In a 'static' setting of an externally prescribed acoustic wave, the required acoustic threshold for instability turns out to be quite high (usually two to three times of the laminar flame speed; see Searby & Rochwerger 1991). However, when the dynamical process of spontaneous sound generation is taken into account, it is found here that there is no need to introduce an acoustic wave of that magnitude to make the flame parametrically unstable. Indeed, the parametric instability is triggered without being forced by any external acoustic wave at all.

The present result indicates that under the action of small inevitable disturbances such as vortical perturbations, a stable planar flame without any acoustic field may evolve into a fully nonlinear regime, where a strong acoustic field is spontaneously produced by the flame. This scenario is probably generic. Mathematically and physically, it may well be possible that two (or more) solutions coexist at the same parameters: a low-branch solution for which the acoustic field is absent or sufficiently weak to remain dynamically passive and an upper-branch solution sustained by an active closed-loop interaction between the flame and its acoustics. Experiments (e.g. Searby 1992) and recent numerical simulations based on the Navier–Stokes equations for compressible reactive flows (Petchenko *et al.* 2006) all point to the existence of a highly curved flame with a self-sustained acoustic field (i.e. an upper-branch solution) in addition to the planar flame (i.e. a lower-branch solution). Indeed, acoustically unstable combustion may correspond precisely to such an upper-branch solution. The present finding for a planar flame suggests that relatively weak ambient perturbations might provoke a switch to a possible upper-branch solution.

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Appendix A. Parametric instability equation with $O(\delta)$ accuracy

The leading-order formulation of the parametric instability can be extended to second order with $O(\delta)$ accuracy by including the $O(\delta)$ jumps as well as viscous diffusion terms in the linearized hydrodynamic equations (2.37). The solution for $(\tilde{U}, \tilde{\boldsymbol{V}}, \tilde{P})$ may still be written as (5.2) with (5.3)–(5.5), but the functions C_+ and $\boldsymbol{D}_+$ are now replaced by $C^+(\xi, t)$ and $\boldsymbol{D}^+(\xi, t)$, which satisfy

$$C^+_\xi + \mathrm{i}\,\boldsymbol{k}^\dagger \cdot \boldsymbol{D}^+ = 0, \qquad (\mathrm{A}\,1)$$

$$\left\{R_+\frac{\partial}{\partial t} + \frac{\partial}{\partial \xi} - \delta Pr\left(\frac{\partial^2}{\partial \xi^2} - k^2\right)\right\}\begin{pmatrix} C^+ \\ \boldsymbol{D}^+ \end{pmatrix} = 0. \qquad (\mathrm{A}\,2)$$

Equation (A 2) can easily be solved by taking Laplace transform with respect to $\xi > 0$. For C^+, its Laplace transform $\widehat{C}^+(s, t)$ satisfies

$$R_+\frac{\partial \widehat{C}^+}{\partial t} + [s - \delta Pr(s^2 - k^2)]\widehat{C}^+ = (1 - \delta Prs)C^+(0^+, t) - \delta Pr C^+_\xi(0^+, t),$$

where $C^+(0^+, t)$ and $C^+_\xi(0^+, t)$ stand for the values of C^+ and C^+_ξ on the burned side of the flame and will be expressed in terms of $\phi_\pm$ by using the jump conditions across the flame (see below). The solution is found to be

$$\widehat{C}^+(s, t) = \frac{R_+^{-1}}{\Lambda(t)}\int_{-\infty}^{t}\{(1 - \delta Prs)C^+(0^+, \tilde{t}) - \delta Pr C^+_\xi(0^+, \tilde{t})\}\Lambda(\tilde{t})\,\mathrm{d}\tilde{t}, \qquad (\mathrm{A}\,3)$$

where we have put

$$\Lambda(t) = \exp\{R_+^{-1}[s - \delta Pr(s^2 - k^2)]t\}.$$

To aid the later calculation, we rewrite (A 3) by integration by parts as

$$\widehat{C}^+(s,t) = -\frac{s^{-1}R_+^{-1}}{\Lambda(t)}\int_{-\infty}^{t}\left\{R_+C_{\tilde{t}}^+(0^+,\tilde{t}) + \delta Prk^2C^+(0^+,\tilde{t}) + \delta PrsC_\xi^+(0^+,\tilde{t})\right\}\Lambda(\tilde{t})\mathrm{d}\tilde{t} + s^{-1}C^+(0^+,t). \quad \text{(A 4)}$$

Similarly,

$$\widehat{\boldsymbol{D}}^+(s,t) = \frac{R_+^{-1}}{\Lambda(t)}\int_{-\infty}^{t}\{[1-\delta Prs]\boldsymbol{D}^+(0^+,\tilde{t}) - \delta Pr\boldsymbol{D}_\xi^+(0^+,\tilde{t})\}\Lambda(\tilde{t})\mathrm{d}\tilde{t}. \quad \text{(A 5)}$$

The continuity equation is Laplace transformed to yield

$$s\widehat{C}^+(s,t) - C^+(0^+,t) + \mathrm{i}\,\boldsymbol{k}^\dagger\cdot\widehat{\boldsymbol{D}}^+ = 0. \quad \text{(A 6)}$$

Inserting (A 4) and (A 5) into (A 6) and making use of (A 1), we arrive at the relation

$$-R_+C_t^+(0^+,t) - \delta Prk^2C^+(0^+,t) + \mathrm{i}\,\boldsymbol{k}^\dagger\cdot\boldsymbol{D}^+(0^+,t) - \delta Pr\,\mathrm{i}\,\boldsymbol{k}^\dagger\cdot\boldsymbol{D}_\xi^+(0^+,t) = 0. \quad \text{(A 7)}$$

The jump conditions, (2.41) and (2.43), imply that

$$C^+(0^+,t) = -(\phi_+ - \phi_-) + \delta\left(\frac{1}{2}lq\mathscr{D}\right)(k^2\alpha + k\phi_- - \mathrm{i}\,\boldsymbol{k}^\dagger\cdot\boldsymbol{D}_-\,\mathrm{e}^{-\mathrm{i}\omega t}) + C_-\,\mathrm{e}^{-\mathrm{i}\omega t}, \quad \text{(A 8)}$$

$$\boldsymbol{D}^+(0^+,t) = \frac{\mathrm{i}\,\boldsymbol{k}^\dagger}{k}(\phi_+ + \phi_-) - q\,(\mathrm{i}\,\boldsymbol{k}^\dagger)\alpha + \boldsymbol{D}_-\,\mathrm{e}^{-\mathrm{i}\omega t} + \delta\left\{Pr\left[\left[\frac{\partial\boldsymbol{V}}{\partial\xi}\right]\right] + \left[\mathrm{i}\,\boldsymbol{k}^\dagger\left(\frac{1}{k}\phi_-'(t) + \alpha' + (G + u_{a,t}(0^-,t))\alpha\right) - \mathrm{i}\,\omega\boldsymbol{D}_-\,\mathrm{e}^{-\mathrm{i}\omega t}\right]\ln(1+q)\right\}, \quad \text{(A 9)}$$

where a direct calculation shows that

$$\boldsymbol{D}_\xi^+(0^+,t) = \mathrm{i}\,\mathscr{S}_-\boldsymbol{D}_-\,\mathrm{e}^{-\mathrm{i}\omega t} - \mathrm{i}\,\boldsymbol{k}^\dagger(\phi_+ - \phi_-) + \left[\left[\frac{\partial\boldsymbol{V}}{\partial\xi}\right]\right]. \quad \text{(A 10)}$$

The above relations are inserted into (A 7) to give

$$\begin{aligned} &R_+(\phi_+' - \phi_-') - k(\phi_+ + \phi_-) + qk^2\alpha - \delta(G + u_{a,t}(0^-,t))\ln(1+q)k^2\alpha \\ &= \delta\left\{\left[\ln(1+q) + \frac{q\,l\mathscr{D}}{2(1+q)}\right](k^2\alpha' + k\phi_-' - \omega(\boldsymbol{k}^\dagger\cdot\boldsymbol{D}_-)\,\mathrm{e}^{-\mathrm{i}\omega t})\right\} \\ &\quad + \{-\mathrm{i}(R_+\omega C_- + \boldsymbol{k}^\dagger\cdot\boldsymbol{D}_-) + \delta[Pr(k^2C_- - \mathscr{S}_-\boldsymbol{k}^\dagger\cdot\boldsymbol{D}_-)]\}\,\mathrm{e}^{-\mathrm{i}\omega t}. \end{aligned} \quad \text{(A 11)}$$

The pressure jump, including the $O(\delta)$ terms, can be written as

$$\begin{aligned} &R_+\phi_+' + R_-\phi_-' - k(\phi_+ - \phi_-) - k\{(R_+ - R_-)G - \Delta p_{a,\xi}\}\alpha \\ &= \delta k\left\{-ql\mathscr{D}(k^2\alpha + k\phi_- - \mathrm{i}\,\boldsymbol{k}^\dagger\cdot\boldsymbol{D}_-\,\mathrm{e}^{-\mathrm{i}\omega t}) - qk^2\alpha + (\phi_-' - \mathrm{i}\,\omega C_-\,\mathrm{e}^{-\mathrm{i}\omega t})\ln(1+q)\right\}. \end{aligned} \quad \text{(A 12)}$$

Subtracting (A 11) from (A 12) leads to

$$\begin{aligned} &(R_+ + R_-)\phi_-' + 2k\phi_- - \{qk^2 + k[(R_+ - R_-)G - \Delta p_{a,\xi}] - \delta k^2(G + u_{a,t}(0^-,t))\ln(1+q)\}\alpha \\ &= -\delta\left\{ql\mathscr{D}(k^3\alpha + k^2\phi_-) + qk^3\alpha + \left[\ln(1+q) + \frac{ql\mathscr{D}}{2(1+q)}\right]k^2\alpha' + \frac{ql\mathscr{D}}{2(1+q)}k\phi_-'\right\} \end{aligned}$$

$$+\left\{ \mathrm{i}(R_+\omega C_- + \boldsymbol{k}^\dagger \cdot \boldsymbol{D}_-) + \delta \left[\left[\ln(1+q) + \frac{q\, l\mathscr{D}}{2(1+q)} \right] \omega(\boldsymbol{k}^\dagger \cdot \boldsymbol{D}_-) \right.\right.$$
$$\left.\left. + (ql\mathscr{D})\,\mathrm{i}\,k\,\boldsymbol{k}^\dagger \cdot \boldsymbol{D}_- - \mathrm{i}\,\omega k C_- \ln(1+q) - Pr(k^2 C_- - \mathscr{S}_- \boldsymbol{k}^\dagger \cdot \boldsymbol{D}_-) \right] \right\} \mathrm{e}^{-\mathrm{i}\,\omega t}. \tag{A 13}$$

Combining this equation with the front equation

$$\alpha' = \phi_- + C_- \mathrm{e}^{-\mathrm{i}\omega t} - \delta \left[\frac{1+q}{q} \ln(1+q) + \frac{1}{2} l\mathscr{D} \right] (k^2\alpha + k\phi_- - \mathrm{i}\,\boldsymbol{k}^\dagger \cdot \boldsymbol{D}_- \mathrm{e}^{-\mathrm{i}\omega t}) \tag{A 14}$$

to eliminate ϕ_- and using (5.22) and (5.23) for $u_{a,t}(0^-, t)$ and $\Delta p_{a,\xi}$, we finally obtain

$$\mathscr{A}\alpha'' + \mathscr{B}\alpha' + [\mathscr{C} - k(\tilde{\chi}_c B(\tau)\,\mathrm{e}^{2\,\mathrm{i}\,\omega t} + \mathrm{c.c.})]\alpha = \mathscr{N}\,\mathrm{e}^{-\mathrm{i}\,\omega t} + \mathrm{c.c.}, \tag{A 15}$$

where the constants $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are given by (3.11) and

$$\tilde{\chi}_c = \chi_c(1 + (\mathrm{i}\,\omega l\mathscr{D})\delta)\left(1 + \delta\frac{1+q}{q}\ln(1+q)k\right),$$
$$\mathscr{N} = 2(k - \mathrm{i}\,\omega R_-)(1 - \mathrm{i}\,\delta Q_F)C_-. \tag{A 16}$$

Again there is no dependence of the Prandtl number, Pr. Equation (A 15) extends the parametric instability equation (5.24) to $O(\delta)$ accuracy and interestingly is of the same form as the leading-order one. Although it has been used by Searby & Rochwerger (1991) and Bychkov (1999), among others, to analyse parametric instability caused by an externally imposed (as apposed to a self-generated) pressure, it does not appear to have been derived before.

Finally, one should bear in mind that an $O(\epsilon)$ error is present due to neglecting the nonlinear terms in the hydrodynamic equations and jumps. As was remarked in § 2, this nonlinear correction can be accounted for by means of a regular perturbation procedure, leading to a further improved theory with $O(\epsilon)$ accuracy, which may be used to check the accuracy of the results obtained in the present paper.

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